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**A New Characterization of the Maximum Cut in a Graph. Dedicated to the memory of the Tibetan meditation master Geshe1 Langri Tangpa (1054-1123), author of the "Eight verses for training the mind"**

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**Institute of Computer Science**  
**Academy of Sciences of the Czech Republic**

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*Dedicated to the memory of the Tibetan meditation master  
Geshe Langri Tangpa (1054-1123), author of the  
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Technical report No. V-1155

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## A New Characterization of the Maximum Cut in a Graph

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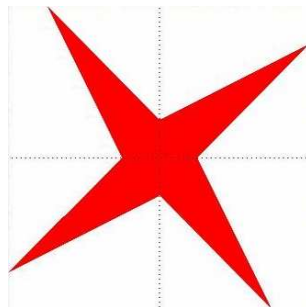
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Abstract:

We prove a formula expressing the maximal cut in a graph in terms of solvability of a system of linear inequalities  $-e \leq Ax \leq e$  ( $e$  being the vector of all ones) appended with a nonlinear constraint  $\|x\|_1 \geq 1$ .<sup>2</sup>



Keywords:

Graph, maximum cut, linear inequalities, norm.

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<sup>1</sup>Equivalent to our “Dr”.

<sup>2</sup>Above: logo of interval computations and related areas (depiction of the solution set of the system  $[2, 4]x_1 + [-2, 1]x_2 = [-2, 2]$ ,  $[-1, 2]x_1 + [2, 4]x_2 = [-2, 2]$  (Barth and Nuding [1])).

## 1 Introduction

Maximum cut in a graph is a well known NP-complete problem. In the main result of this report (Theorem 1) we prove a formula expressing the maximal cut in a graph in terms of solvability of a system of linear inequalities

$$-e \leq Ax \leq e$$

( $e$  being the vector of all ones) appended with a nonlinear constraint

$$\|x\|_1 \geq 1.$$

In this way the original discrete problem is recast as a continuous weakly nonlinear problem which can be solved by nonlinear optimization techniques. A related decision problem of determining whether the maximum cut exceeds a prescribed nonnegative integer  $\ell$  is handled in Corollary 3.

## 2 Maximum cut: definition

Let  $G = (N, E)$  be an undirected graph with set of nodes  $N = \{1, \dots, n\}$  and set of edges  $E$ . Let  $m$  denote the cardinality of  $E$ .

Let  $A_G = (a_{ij})$  be given by  $a_{ij} = n$  if  $i = j$ ,  $a_{ij} = -1$  if  $i \neq j$  and the nodes  $i, j$  are connected by an edge, and  $a_{ij} = 0$  if  $i \neq j$  and  $i, j$  are not connected. Then  $A_G$  is an MC-matrix [4].

For  $S \subseteq N$ , define the cut  $c(S)$  as the number of edges in  $E$  whose one endpoint belongs to  $S$  and the other one to  $N \setminus S$ . Then the maximum cut in  $G$  is defined by

$$\text{mc}(G) = \max_{S \subseteq N} c(S).$$

Computation of the maximum cut in a graph is known to be an NP-complete problem [2].

## 3 Maximum cut: characterization

We denote  $\mathcal{N} = \{0, 1, 2, \dots\}$  (the set of nonnegative integers),  $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ , and we use the norm  $\|x\|_1 = e^T |x| = \sum_{i=1}^n |x_i|$ . Then we have this characterization which is the main result of this report.

**Theorem 1.** *For each undirected graph  $G$  there holds*

$$\text{mc}(G) = \max\{\ell \in \mathcal{N} \mid -e \leq (4\ell - 2m + n^2)A_G^{-1}x \leq e, \|x\|_1 \geq 1 \text{ has a solution}\}.$$

*Proof.* The result follows from the relation

$$\text{mc}(G) = \frac{1}{4} \left( \max_{z \in \{-1, 1\}^n} z^T A_G z + 2m - n^2 \right)$$

established in the proof of Theorem 3 in [4] and from Proposition 3 in [3]. □

It remains to be shown how a maximum cut  $c(S)$  can be found.

**Theorem 2.** *Let  $x$  be any solution of the system*

$$-e \leq (4\text{mc}(G) - 2m + n^2)A_G^{-1}x \leq e,$$

$$\|x\|_1 \geq 1.$$

*Then the set*

$$S = \{i \mid x_i \geq 0\}$$

*satisfies*

$$c(S) = \text{mc}(G).$$

*Proof.* This description is a consequence of construction made in the proof of Theorem 3 in [4].  $\square$

## 4 Maximum cut: lower bounds

As immediate consequences of Theorems 1 and 2 we obtain these two corollaries.

**Corollary 3.** *Let  $G$  be an undirected graph and  $\ell$  a nonnegative integer. Then*

$$\text{mc}(G) \geq \ell \tag{4.1}$$

*holds if and only if the system*

$$-e \leq (4\ell - 2m + n^2)A_G^{-1}x \leq e, \tag{4.2}$$

$$\|x\|_1 \geq 1 \tag{4.3}$$

*has a solution.*

**Corollary 4.** *If the system (4.2), (4.3), where  $\ell$  is a nonnegative integer, has a solution  $x$ , then the set*

$$S = \{i \mid x_i \geq 0\}$$

*satisfies*

$$c(S) \geq \ell.$$

*If (4.2), (4.3) has no solution, then*

$$\text{mc}(G) < \ell.$$

## 5 Maximum cut: algorithm

Corollary 3 shows us a way how to verify (or disprove) the inequality (4.1) via solving a system of inequalities of the type

$$-e \leq Ax \leq e, \tag{5.1}$$

$$\|x\|_1 \geq 1. \tag{5.2}$$

Such an algorithm, named **basintnprob** [from BASic INTerval NP PROBLEM], was described in [5]. As proved there, the algorithm in a finite number of steps either finds a solution to (5.1), (5.2), or states that no such solution exists.

## Bibliography

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