On Probabilistic Transformations of Belief Functions

Daniel, Milan
2005

Institute of Computer Science
Academy of Sciences of the Czech Republic

On Probabilistic Transformations of Belief Functions

Milan Daniel

Technical report No. 934

May 2005
Revised September 2005

Pod Vodárenskou věží 2, 182 07 Prague 8, phone: (+420) 266 052 083, (+420) 286 585 789,
e-mail: ics@cs.cas.cz, www home page: http://www.cs.cas.cz

Institute of Computer Science
Academy of Sciences of the Czech Republic

On Probabilistic Transformations of Belief Functions [1]

Milan Daniel[2]

Technical report No. 934

May 2005
Revised September 2005

Abstract:

Alternative approaches to the widely known pignistic transformation of belief functions are presented and analyzed. A series of various probabilistic transformations is examined namely from the point of view of their consistency with rules for belief function combination and their consistency with probabilistic upper and lower bounds. A new definition of general probabilistic transformation is introduced and a discussion of their applicability is included.

Keywords:
Belief function, Combination of belief functions, Dempster-Shafer theory, Probabilistic transformation, Pignistic transformation, Pignistic probability, Combination consistency, ulb-consistency.

[1] Partial support by the COST action 274 TARSKI acknowledged.
The work was partly supported by the Institutional Research Plan AV0Z10300504 "Computer Science for the Information Society: Models, Algorithms, Applications".
[2] milan.daniel@cs.cas.cz

1 Introduction

Belief functions are formalisms widely used for uncertainty representation and processing.

For combination of beliefs the Dempster's rule of combinations is used in DST. Under strict probabilistic assumptions its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. There are not rare examples where the assumptions are not fulfilled and where results of the Dempster's rule are counter intuitive, e.g. see [4, 5], thus a rule with more intuitive results is required in such situations.

Hence series of modifications of the Dempster's rule were suggested and alternative approaches were created. The classical ones are the Dubois-Prade's rule [21] and the Yager's belief combination rule [40]. Among the others a wide class of weighted operators [29], Transferable Belief Model (TBM) using so called non-normalized Dempster's rule [32, 33], disjunctive (or dual Demspter's) rule of combination [9, 20], combination 'per elements' [6] with its special case — minC combination, see [8], subjective logic with Consensus Operator [28], and other combination rules. It is also necessary to mention the method for application of the Dempster's rule in the case of partially reliable input beliefs [23].

Subsequently, numerous practical applications were suggested and implemented in a wide range of domains.

What is common for their applications? It is an aim to transform the resulting evidence representation by a general belief function to representation by probability for the purpose of easier decision making, resulting beliefs comparison and ordering. Such a probability should be consistent with the original belief function. In fact, we can consider it as a belief function of a special type, so called Bayesian belief function. We call such a transformation as a *probabilistic transformation*.

Frequently only a special case of probabilistic transformation – Pignistic transformation — is used. In the last years several papers on alternative probabilistic transformations have been published [2, 3, 12, 13, 36, 37], and a new justification of pignistic transformation has appeared [34, 35].

This report summarizes and completes the study of probabilistic transformations presented in [12, 13, 15]. Besides the new original results, Baroni & Vicigs's results from [2] and Cobb & Shenoy's results [3], the present study includes also Sudano's transformations [36, 37] and Smets' new results [34, 35].

Basic notions, both general, and those from [12] and [13] are introduced in Section 2. Section 3 presents a series of probabilistic transformations from various sources and it shows that some of them are equivalent to other one(s). Section 4 brings a summary of consistencies of the transformations. A new definition of the general probabilistic transformation based on their analysis and a justification of two main alternatives to pignistic transformation is presented in Section 5. A discussion about which transformation should be applied in applications concludes the study.

2 Preliminaries

2.1 Basic notions

Let us first recall some basic notions from the theory of belief functions. Let us consider an n-element frame of discernment[3] $\Omega = \{\omega_1, \omega_2, ...\omega_n\}$. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. If $m(\emptyset) = 0$, we speak about *normalized bba*. A *belief function (BF)* is a mapping $bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. $\mathcal{P}(\Omega)$ is often denoted by 2^{Ω}. Let us further recall a *plausibility function* $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$, a commonality function $Q(A) = \sum_{A \subseteq X \subseteq \Omega} m(X)$, and *doubt function* $Dou(A) = Bel(\overline{A})$.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function*, it is a probability distribution on Ω in fact. If all the focal elements are either singletons or whole Ω

[3] We use the classical Shaferian terminology. Besides, it is also possible to use the new more user-friendly simplification of the terminology suggested by Dempster, see e.g. [16], using a notion *state space* instead of a frame of discernment, and similarly.

(i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a *quasi-Bayesian belief function*, it is something like 'non-normalized probability distribution'.

To underline the cardinality of a frame of discernment, we use the left lower indices, e.g. ${}_{nD}bel(X)$, ${}_{3D}m(X)$, etc., and we speak about nD BF *bel*, 3D bba m, etc. Let ${}_{2D}0 = (0,0)$ and ${}_{nD}0 = (0,...,0)$ denote special BFs bel_0 such that $m_0(\Omega) = 1$, ${}_{2D}0' = (\frac{1}{2}, \frac{1}{2})$ and ${}_{nD}0' = (\frac{1}{n}, ..., \frac{1}{n}, 0, ..., 0)$ denote special BFs $bel_{0'}$ such that $m_{0'}(X) = \frac{1}{n}$ for $|X| = 1$.

The Dempster's (conjunctive) rule of combination is given as $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, $A, X, Y \subseteq \Omega$, where $K = 1/(1 - \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)) = \frac{1}{1-\kappa}$, and $m(\emptyset) = 0$, see [30]; putting $K = 1$ and $m(\emptyset) = \kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e.g. [33]. *The disjunctive rule of combination* is given by the formula $(m_1 \circledcirc m_2)(A) = \sum_{X \cup Y = A} m_1(X) m_2(Y)$, see [22]. Specially for $(m_1(1), m_1(0)) = (a, b)$, $(m_2(1), m_2(0)) = (c, d)$ on $\Omega = \{0, 1\}$, we have $(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)})$ and $(a, b) \circledcirc (c, d) = (ac, bd)$.

Bayes' rule of probability combination is defined as a normalized point-wise multiplication of probabilities of singletons. $(P_1 \otimes P_2)(x) = \frac{P_1(x) P_2(x)}{\sum_{y \in \Omega} P_1(y) P_2(y)}$.

The Yager's rule of combination $\textcircled{Y}$, see [40], is defined as $(m_1 \textcircled{Y} m_2)(A) = \sum_{X,Y \subseteq \Omega,\, X \cap Y = A} m_1(X) m_2(Y)$ for $\emptyset \neq A \subset \Omega$, $(m_1 \textcircled{Y} m_2)(\Omega) = m_1(\Omega) m_2(\Omega) + \sum_{X,Y \subseteq \Omega,\, X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \textcircled{Y} m_2)(\emptyset) = 0$.

The Dubois-Prade's rule of combination $\textcircled{D}$ is defined for normalized BFs by the formula $(m_1 \textcircled{D} m_2)(A) = \sum_{X,Y \subseteq \Omega,\, X \cap Y = A} m_1(X) m_2(Y) + \sum_{X,Y \subseteq \Omega,\, X \cap Y = \emptyset, X \cup Y = A}$ for $\emptyset \neq A \subseteq \Omega$, and $(m_1 \textcircled{D} m_2)(\emptyset) = 0$, see [21].

A *consensus operator* $\textcircled{c}$ is defined as $(b_1, d_1) \textcircled{c} (b_2, d_2) = (\frac{b_1 u_2 + b_2 u_1}{u_1 + u_2 - u_1 u_2}, \frac{d_1 u_2 + d_2 u_1}{u_1 + u_2 - u_1 u_2})$, where $0 \leq u_i = 1 - b_i - d_i \leq 1$ for $u_1 + u_2 \neq 0$, $(b_1, d_1) \textcircled{c} (b_2, d_2) = (\frac{b_1 + b_2}{2}, \frac{d_1 + d_2}{2})$.

If we group some elements of Ω_0 to disjoint groups not further distinguishing their members we speak about *coarsening* of Ω_0 to Ω. We have $m(\{X\}) = bel_0(X)$ for $X \in \Omega$ and $m(\{X\}) = bel_0(X) - \sum_{\emptyset \neq Y \subset X} m(Y)$ for $X \subseteq \Omega$. On the other hand, when dividing some element(s) of Ω into several disjoint ones, we speak about *refinement* of the frame of discernment, e.g. the above Ω_0 is a refinement of Ω. We have $m_0(\{X\}) = m(X) - \sum_{Y \subset X} m_0(Y)$ for $X \subseteq \Omega$.

2.2 Notions related to algebraic analysis

Let us recall the following terminology and denotation from algebraic analyses of belief functions and belief combination on 2-element frame of discernment, see e.g. [9, 10, 25, 26]. Let us assume $\Omega = \{\omega_1, \omega_2\}$. It holds that ${}_{2D}m(\omega_1) + {}_{2D}m(\omega_2) + {}_{2D}m(\Omega) = 1$, thus ${}_{2D}m(\Omega) = 1 - ({}_{2D}m(\omega_1) + {}_{2D}m(\omega_2))$, hence every 2D BF *bel* uniquely corresponds to the pair $(m(\omega_1), m(\omega_2))$. Such a pair is called a *Dempster's pair* or briefly *d-pair*. Extremal d-pairs are the pairs corresponding to BFs for which either $m(\omega_1) = 1$ or $m(\omega_2) = 1$. The set of all non-extremal d-pairs is denoted as D_0, the set of all non-extremal Bayesian d-pairs (i.e. d-pairs corresponding to 2D Bayesian BFs, where $m(\omega_1) + m(\omega_2) = 1$) is denoted as G, the set of d-pairs such that $m(\omega_1) = m(\omega_2)$ is denoted as S, and the set where $m(\omega_2) = 0$ as S_1, analogically $m(\omega_1) = 0$ in S_2. For the special belief functions 0 and $0'$ it holds that $m(\omega_1) = m(\omega_2) = 0$, resp. $\frac{1}{2}$, i.e. ${}_{2D}0 = (0,0)$ and ${}_{2D}0' = (\frac{1}{2}, \frac{1}{2})$. Analogically we can define ${}_{nD}0 = (0, 0, ..., 0)$ and ${}_{nD}0'$ as $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}, 0, ..., 0)$, where we have bbm $\frac{1}{n}$ for singletons and 0 for non-singletons.

The *(standard/conjunctive) Dempster's semigroup* $\mathbf{D}_0 = (D_0, \oplus)$ is the set of all non-extremal Dempster's pairs, endowed with the binary operation $\oplus$ (i.e. with the Dempster's rule) and two distinguished elements $0 = (0,0)$ and $0' = (\frac{1}{2}, \frac{1}{2})$. Let $D_{\circledcirc}$ denote $D_0 \cup \{(1,0), (0,1), (1,1)\}$. The *disjunctive Dempster's semigroup* $\mathbf{D}_{\circledcirc} = (D_{\circledcirc}, \circledcirc)$ is the set of all Dempster's pairs extended by $1 = (1,1)$, endowed with the operation $\circledcirc$ and two distinguished elements $0 = (0,0)$ and $1 = (1,1)$.

A *homomorphism* $p : (X, \otimes_1) \longrightarrow (Y, \otimes_2)$ is a mapping which preserves structure, i.e. $p(x \otimes_1 y) = p(x) \otimes_2 p(y)$ for each $x, y \in X$. We will use the following homomorphisms:
$h : (D_0, \oplus) \longrightarrow (G, \oplus)$, $h(bel) = bel \oplus 0'$, $h(a,b) = (a,b) \oplus 0' = (\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b})$, and its nD generalization $h(bel) = bel \oplus {}_{nD}0'$, see [14],
$u : (D_0, \circledcirc) \longrightarrow (G, \circledcirc \circ u)$, $u(a,b) = (a,b) \circledcirc (\frac{1}{a+b}, \frac{1}{a+b}) = (\frac{a}{a+b}, \frac{b}{a+b})$, and its nD generalization

$u(x_1, ..., x_n, x_{n+1}, ..., x_{2^n-1}) = (\frac{x_1}{\sum_{i=1}^n x_i}, ..., \frac{x_n}{\sum_{i=1}^n x_i}, 0, ..., 0)$, see [14].

2.3 General definition of probabilistic transformations

Let us consider the following very general definition now [4]. A *probabilistic transformation* (or briefly a *probabilization*) is a mapping $T : Bel_\Omega \longrightarrow ProbDistr_\Omega$, Thus the probabilistic transformation assigns a Bayesian belief function (i.e. probability distribution) to every general one. It is a reason why the transformations of belief functions to probability distributions are sometimes called also Bayesian transformations, see e.g. [38]. As we suppose finite frames of discernments, we can compute $(T(bel))(X) = \sum_{A \in X}(T(bel))(A)$ for any $X \subseteq \Omega$.

The fundamental well know example of a probabilistic transformation is the pignistic transformation $BetT$ and its resulting pignistic probability $BetP$ [5] introduced by Smets. We do not use the name pignistic transformation for the other ones, and we use the general name probabilistic transformation, in accordance with Philippe Smets' wish not to mix new alternatives together with his classical pignistic transformation. Moreover, it allows us to use a more general definition with less assumptions.

2.4 ulb-consistency and p-consistency

Probabilistic transformation PT is *ulb-consistent (upper and lower bound consistent)* if its resulting transformed probability TP satisfies the following consistency condition: $Bel(X) \leq TP(X) \leq Pl(X) = 1 - Bel(\overline{X})$. Probabilistic transformation PT is *p-consistent (or probabilistically consistent)* if $PT(m) = m$ for any Bayesian bba m. In other words Bayesian BFs are fix points of p-consistent PTs. p-consistency is in fact ulb-consistency on Bayesian BFs (i.e. weakening of ulb-consistency) because $bel(X) = Pl(X)$ for Bayesian BFs.

2.5 Combination consistencies

A combination consistency of a PT is based on commutation of a combination rule $\circledast$ with PT, i.e. we obtain the same results if we combine beliefs bel_1 and bel_2 using the combination rule $\circledast$ and perform PT after it as in the case, where we first compute probabilistic transformations of the both input beliefs bel_1 and bel_2 and combine them with the combination rule $\circledast$ after.

Probabilistic transformation PT is $\oplus$*-consistent* if it commutes with the Dempster's rule (with $\oplus$ combination). Analogically [6] PT is $\circledcirc$-consistent if it commutes with $\circledcirc \circ u$. Where u stands for the nD generalization of the original 2D homomorphism u: ${}_{2D}u(a, b) = (a, b)\circledcirc(\frac{1}{a+b}, \frac{1}{a+b}) = (\frac{a}{a+b}, \frac{b}{a+b})$, and its nD generalization $u(x_1, ..., x_n, x_{n+1}, ..., x_{2^n-1}) = (\frac{x_1}{\sum_{i=1}^n x_i}, ..., \frac{x_n}{\sum_{i=1}^n x_i}, 0, ..., 0)$, see [9, 14].

Probabilistic transformation PT is $\oplus_I$*-consistent* if it keeps (it is closed to) $\oplus$-idempotents, i.e. $PT(m_i) \oplus PT(m_i) = PT(m_i)$ for all m_i such that $m_i \oplus m_i = m_i$. It is easy to observe, that $\oplus_I$-consistency is a weaker version of $\oplus$-consitency. Analogically we define $\circledcirc_I$*-consistency* weaker versions of $\circledcirc$-consistency.

3 Probabilistic transformations

3.1 Pignistic transformation

The *pignistic transformation* $BetT$ distributes $m(X)$ equally among all elements of X. It was named and justified by Smets in [31] for Transferable Belief Model (TBM), see [31, 33] in 1990. Nevertheless, the transformation based on the same principle was used by Dubois & Prade [19] as "equidistribution of the values of bba" and by Williams [39] in 1982 already.

[4] For a precision of the definition see Section 5.

[5] We denote all transformations with suffix T and related probabilities with P.

[6] It is possible to define analogically other combination consistencies w.r.t. to other combination rules, see e.g. $\circledcirc$-*consistency* [13]. Due to the limitation of applicability of the *consensus operator* $\circledcirc$ [10, 28] only to quasi-Bayesian BFs [11], we omit a presentation of $\circledcirc$-consistency in this text.

The pignistic transformation $BetT$ projects BF bel given bba m to probability $BetP$ defined on the frame of discernment Ω as follows:

$$BetP(A) = \sum_{A \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1 - m(\emptyset)}.$$

It includes normalization and division of bbms assigned to focal elements by their cardinality, non-normalized beliefs used in TBM are admissible.

The original justification of the pignistic transformation [31, 33] unfortunately refers the *principle of insufficient reason.* This disputable principle, which brings some contradictions in the probability theory, has been replaced by the so called *linearity property* assumption in the recent publications, see e. g. [34, 35], i. e. with commutation of the transformation with a convex combination of beliefs: $T(\alpha m_1 + (1-\alpha) m_2) = \alpha T(m_1) + (1-\alpha) T(m_2)$. [7] In correspondence with the definition of combination consistencies we can call the linearity property assumption as *α-consistency.* No justification of the transformation has been presented by Dubois & Prade or by Williams.

From the definition and justification of the pignistic transformation, we can immediately see that it is *ulb*-consistent and α-consistent. BetT is neither $\oplus$-consistent, $\oplus_I$-consistent, nor $\circledcirc$-consistent. BetT is $\oplus_I$-consistent for all nD quasi-Bayesian BFs and consequently for all 2D BFs, further it is $\circledcirc_I$-consistent on the set of all general nD BFs.

3.2 Plausibility or cautious probabilistic transformation

Let us introduce four different definitions of the main alternative to pignistic probability in this subsection.

Widely known it the following one. The *(normalized) plausibility probabilistic transformation* Pl_T, see e.g. [2] or [3], is defined as a normalized plausibility of singletons [8]. Hence we have

$$Pl\_P(A) = \frac{Pl(A)}{\sum_{B \in \Omega} Pl(B)} = \frac{\sum_{A \in X \subseteq \Omega} m(X)}{\sum_{B \in \Omega} \sum_{B \in X \subseteq \Omega} m(X)}.$$

This transformation is called 'the pignistic *probability proportional to normalized plausibility*' ($PrNPl$) by Sudano in [37]. [9]

As commonality $Q(A)$ of a singleton A is equal to its plausibility $Pl(A)$, i.e. $Q(A) = Pl(A)$ for all $A \in \Omega$, we can define Pl_T also by a new formula

$$Pl\_P(A) = Q\_P(A) = \frac{Q(A)}{\sum_{B \in \Omega} Q(B)},$$

and call it the *(normalized) commonality probabilistic transformation Q_T.*

The *cautious probabilistic transformation* [12, 15] is defined as the Dempster's combination of a belief bel with $0'$: $CautT(bel) = bel \oplus 0'$. It is a generalization of homomorphism h, which corresponds to Hájek & Valdés results on 2D belief functions [25, 26]: ${}_{2D}CautP(A) = \frac{1-m(B)}{2-m(A)-m(B)}$.[10] In the nD case we have:

$$CautP(A) = \frac{\sum_{A \in X} m(X)}{n - \sum_{B \in \Omega, X \subseteq \Omega, B \notin X} m(X)}.$$

Voorbraak's Bayesian transformation (VBT)[11] published in 1989, see [2] and [38], is given by

$$VBP(A) = \frac{\sum_{A \in X} m(X)}{\sum_{Y \subseteq \Omega} (m(Y) \cdot |Y|)}.$$

[7] The special case of a convex combination of bbas for $\alpha = \frac{1}{2}$ was mentioned as averaging of bbas in [13].

[8] Despite of the fact that, Cobb and Shenoy introduce it as a new method [3] in 2003, and Sudano also introduces it as $PrNPl$ in 2003, it was known already in 1991 [1].

[9] This name does not correspond to Smets' wish of using the name of the pignistic transformation, besides it does not satisfy all assumptions required from Smets' pignistic transformation, either the original [31, 33] or the recent ones [34, 35]. For this reason we eliminate the word 'pignistic' from the name of the transformation and add a letter T (or P) to abbreviation of the transformation (or resulting probability) to obtain $PrNPlT$ (or $PrNPlP$) to be consistent with the other names. The same holds also for the other Sudano's transformations, see [36, 37].

[10] This 2D transformation was used already in the Expert System Shell EQUANT-PC in late 80's, see [24].

[11] Voorbraak proposed VBT not for decision making, but for approximation of BFs.

Theorem 1 *The cautious, plausibility and commonality probabilistic transformations and Voorbraak's Bayesian transformation are the same transformations of belief functions to probabilistic distributions, i.e. it holds that* $CautP(A) = Pl\_P(A) = Q\_P(A) = VBP(A)$.

Proof: For equality $CautT \equiv Pl\_T$ see [15], for equality $Pl\_T \equiv VBT$ see [2], and equality $Q\_T \equiv Pl\_T$ follows the equality $Q(\{\omega_i\}) = Pl(\{\omega_i\})$ for singletons.

The notion *(normalized) commonality probabilistic transformation* is the best name for this transformation, because it corresponds with the nature of the PT better than the others. It follows also the discussion after presentation of this topic in ECSQARU'05, Barcelona, July 7. On the other hand (normalized) plausibility (probabilistic) is the most frequent in literature till now. To be consistent with the text published in proceedings of ECSQARU'05 we keep the name (normalized) plausibility probabilistic transformation Pl_T for this transformation, further in this text.

Pl_T is $\oplus$-consistent. It is neither $\circledcirc$-consistent nor α-consistent. Pl_T is neither *ulb*-consistent in general. It is *ulb*-consistent for quasi-Bayesian BFs only; it implies *p*-consistency in general on nD and *ulb*-consistency on 2D BFs.

3.3 Belief or disjunctive probabilistic transformation

In [12], the *disjunctive probabilistic transformation* $DisjT$ has been presented which has been defined on 2D frames so that it commutes with $\circledcirc \circ u$, $DisjP(\{A\}) = \frac{m(\{A\})}{m(\{A\})+m(\Omega-\{A\})}$. Its nD generalization is given by the following formula, see [15],

$$DisjP(\{A\}) = \frac{m(A)}{\sum_{X\in\Omega} m(X)}.$$

A *(normalized) belief probabilistic transformation* Bel_T [13] is defined as a normalization of beliefs of singletons (bbms of singletons), i.e. by the formula:

$$Bel\_P(A) = \frac{m(A)}{\sum_{X\in\Omega} m(X)}.$$

Because the formulas are the same, it is evident that $Bel\_T \equiv DisjT$. We have to note that Bel_T is not defined if $\sum_{X\in\Omega} m(X) = 0$; we can complete its definition analogically to the proportional transformation, see later, but such a definition breaks the $\circledcirc$-consistency which was a motivation for definition of $DisjT$. Further, we have to note that Bel_T is significantly sensitive to the bbms of singletons because it ignores completely the bbms of non-singleton focal elements.

Bel_T is $\circledcirc$-consistent, it is not $\oplus$-consistent, but it is $\oplus_I$-consistent whenever it is defined. It is neither α-consistent nor *ulb*-consistent in general. Similarly to Pl_T, Bel_T is also *ulb*-consistent only for quasi-Bayesian BFs; it implies *p*-consistency in general on nD and *ulb*-consistency on 2D BFs.

3.4 A simple map example

We can simply graphically represent 2D BFs in the triangle $(0,0)(1,0)(0,1)$, thus we can also represent probabilistic transformations on 2D BFs in such a way. Let us consider the old Dempster's example, see e.g. [25]. Because of the higher illustrativity of the figures bellow, the values are slightly modified in the example. Let us imagine a (digitalized) map showing areas of land and water, with 0.75 of the area of the map being visible and the visible area divided with proportions 0.8 to 0.2 of the water area to the land area. Thus we have $\Omega = \{W(ater), L(and)\}$, and $m(\{W(ater)\}) = 0.8 \cdot 0.75 = 0.60 = w$, $m(\{L(and)\}) = 0.2 \cdot 0.75 = 0.15 = l$, $m(\{W, L\}) = 0.25$. Pignistic, Plausibility and Belief probabilistic transformations of bbm m from the above example are presented in Figure 3.1, we have to recall that $Pl\_T \equiv CautT$, and $Bel_T \equiv PropT$. For a graphical demonstration of the equivalence of Plausibility and Cautious probabilistic transformations for 2D BFs see Figure 3.2.

Figure 3.3 illustrate a comparison of probabilistic transformations of two beliefs from the map example, which are given by bbas m_1 and m_2, such that $m_1(W) = 0.6, m_1(L) = 0.15$ as in the example from the previous section and $m_2(W) = 0.7$ and $m_2(L) = 0.25$. We can easily observe that $BetT(bel_1) = BetT(bel_2) = (0.725, 0, 275)$, and that there is no difference between decisions according to bel_1 and bel_2 if our decision is based on $BetP_i s$, see the Figure.

If we use cautious probabilization $CautT$ and its resulting cautious probabilities $CautP_i$, probabilities $CautP_2$ are closer to $BetT(bel_i)$, the difference between $CautP_2$ and $BetT(bel_i)$ is less and hence the results are less expressive for bel_2. $CautT(0.6, 0.15) = (\frac{85}{125}, \frac{40}{125}) = (0.68, 0.32) < (0.714, 0.286) = (\frac{75}{105} \frac{30}{130}) = CautT(0.7, 0.25)$. (The resulting probabilities are compared according to their first coordinates, i.e. according to $CautP_i(W)$ in our map example, where $CautP_1(W) < CautP_2(W)$).

An analogical but quite different situation arises if we use the proportional probabilistic transformation. In this case we have $PropT(0.6, 0.15) = (\frac{60}{75}, \frac{15}{75}) = (0.8, 0.2) > (0.737, 0.263) = (\frac{70}{95}, \frac{25}{75}) = PropT(0.7, 0.25)$. The difference is also less for bel_2, but, on the other hand, not only the sizes of the probabilities are different but also their ordering is different — reverse in our simple example — $PropP_1(W) > PropP_2(W)$, in the case that a different probabilistic transformation is used.

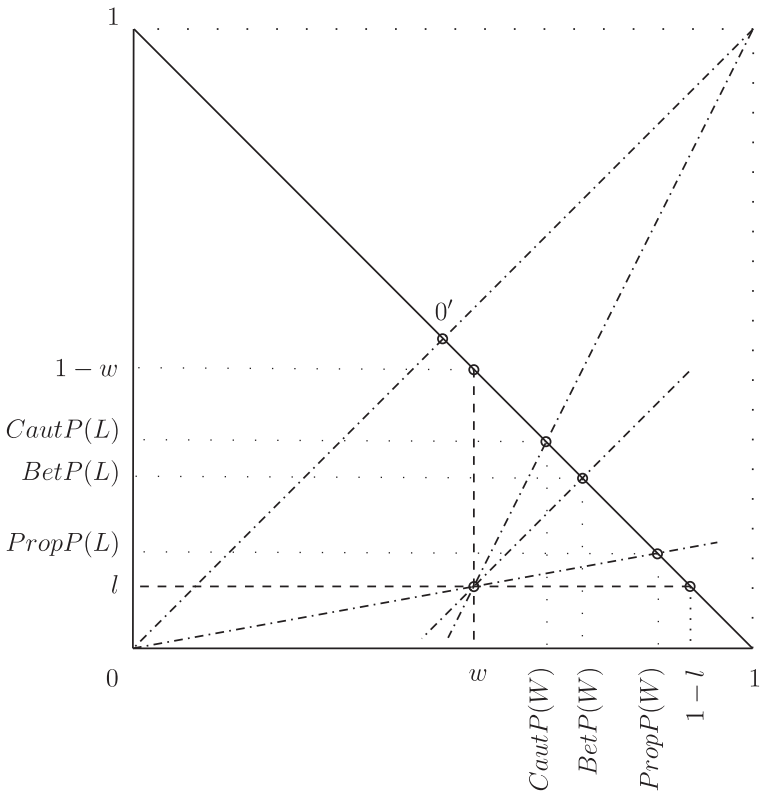

Figure 3.1: ***Comparison of probabilistic transformations*** $BetT$, $Pl\_T \equiv CautT$ ***and*** $Bel_T \equiv PropT$ ***on the Map example.***

3.5 Proportional probabilistic transformations

Proportional transformations take bbm $m(A)$ of a singleton A and add to it proportional parts of $m(X)$ for all its supersets $A \subset X$. From this assumption it is obvious that these *proportional probabilistic transformations* are *ulb*-consistent.

If the proportionalization is computed with respect to the beliefs of singletons, we speak about the *proportional belief probabilistic transformation* $Prop_{Bel}T$, see [13, 15]:

$$Prop_{Bel}P(A) = \sum_{A \in X \subseteq \Omega} \frac{m(A)}{\sum_{B \in X} m(B)} \cdot m(X).$$

If $\sum_{B \in X} m(B) = 0$, then $|X|$ is used instead of it and thus $m(X)$ is relocated per the same portions among all elements of X in such a case.

The equivalent *proportional belief transformation* $PrBlT$, see [36, 37], is based on the same idea as $Prop_{Bel}T$, also the formula for computing of $PrBlP$ corresponds to that for computing $Prop_{Bel}P$.

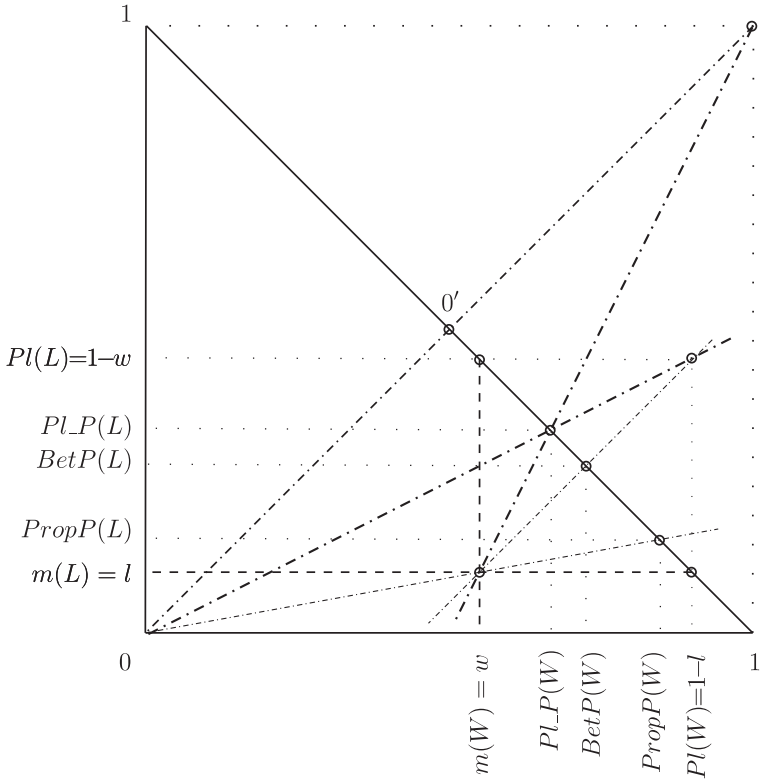

Figure 3.2: ***A comparison of Plausibility and Cautious (probabilistic) transformations.***

Hence $PrBlT \equiv Prop_{Bel}T$.

In order to correct a statement from [13], we have to note that the equivalence $Bel\_T \equiv Prop_{Bel}T$ holds only on 2D and nD quasi-Bayesian BFs only.

$Prop_{Bel}P(A)$ is defined for all BFs, but similarly to Bel_T it is also significantly sensitive to the bbms of singletons. To improve it, the *stepwise proportional belief probabilistic transformation* $StProp_{Bel}T$ or simply *stepwise belief transformation* $StBel\_T$ has been defined in [13]. Bbms $m^{(i-1)}(X)$ for $|X| = (n+1-i)$ are proportionally relocated in the i-th step among $m^{(i)}(Y)$ for $Y \subset X$, $|Y| = (n-i)$. After $(n-1)$ steps all the bbms are finally relocated among singletons. $m^{(0)} = m$, $StBel\_P(A) = m^{(n-1)}(A)$.

$$m^{(i)}(Z) = m^{(i-1)}(Z) + \sum_{\substack{Z \subset X \\ |X|=|Z|+1}} \frac{m^{(i-1)}(Z)}{\sum_{Y \subset X, |Y|=|Z|} m^{(i-1)}(Y)} \cdot m^{(i-1)}(X) \ for \ |Z| = n-i,$$

$m^{(i)}(Z) = m^{(i-1)}(Z) = m(Z)$ for $|Z| < n-i$, and $m^{(i)}(Z) = 0$ for $|Z| > n-i$.
If $\sum_{Y \subset X, |Y|=|X|} m(Y) = 0$ then $|X|$ is used instead of it, thus $m(X)$ is relocated per the same portions among all Y in such a case.

If the proportionalization is computed with respect to the plausibilities of singletons, we speak about the *proportional plausibility probabilistic transformation* $Prop_{Pl}T$, see [13], which is defined by

$$Prop_{Pl}P(A) = \sum_{A \in X \subseteq \Omega} \frac{Pl(A)}{\sum_{B \in X} Pl(B)} \cdot m(X).$$

The equivalent *proportional plausibility transformation* $PrPlT$ [36, 37] is based on the same idea as $Prop_{Pl}T$, also the formula for computing of $PrPl$ corresponds to that for computing $Prop_{Pl}P$. Hence $PrPlT \equiv Prop_{Pl}T$.

If the proportionalization is computed with respect to commonalities of singletons we speak about the *proportional commonality probabilistic transformation* $Prop_QT$

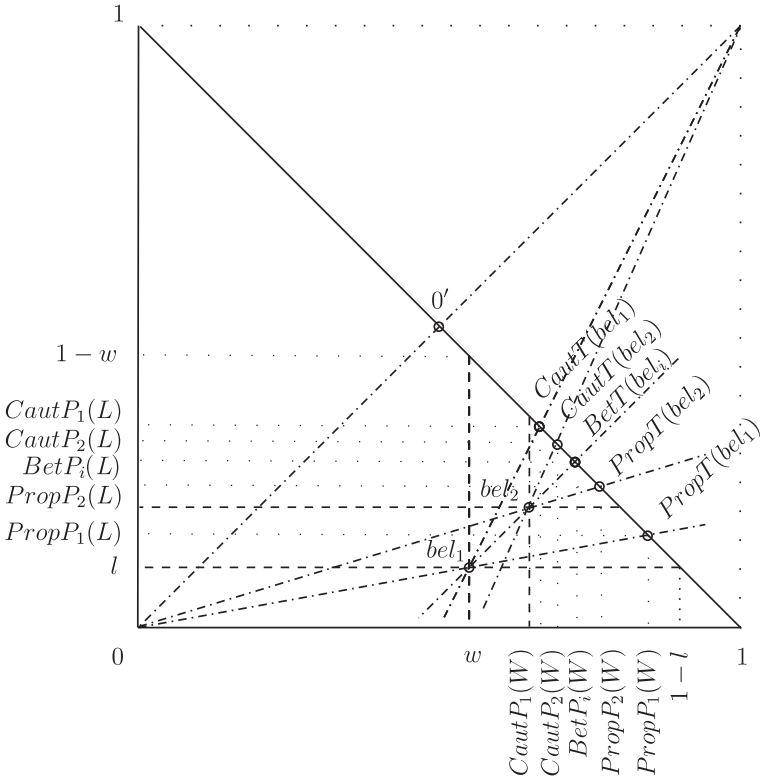

Figure 3.3: ***Comparison of probabilistic transformations of two beliefs from the Map example.***

$$Prop_Q P(A) = \sum_{A \in X \subseteq \Omega} \frac{Q(A)}{\sum_{B \in X} Q(B)} \cdot m(X).$$

It holds that $Prop_Q T \equiv Prop_{Pl} T$ because of equality of plausibilities and commonalities of singletons.

Two other probabilistic proportional transformations are defined by Sudano in [36], see also [37]. *Probability deficiency transformation* $PraPlT$ and iterative *proportional self-consistent probabilistic transformation* $PrScT$.

$$PraPl(A) = m(A) + \frac{1 - \sum_{B \in \Omega} m(B)}{\sum_{B \in \Omega} Pl(B)} \cdot Pl(A).$$

$PraPlT$ is equal to $PrPlT$ and $Prop_{Pl} T$ on 2D and on nD qBBFs, but it does not satisfy our introductive assumption of proportional probabilistic transformations. Moreover, it is not *ulb*-consistent in general, even if its *ulb*-consistency is assumed and claimed in [36] [12]. Nevertheless, $PraPlT$ satisfies the weaker *p*-consistency.

$$PrScP(A) = \sum_{A \in X} \frac{PrScP(A)}{\sum_{B \in X} PrScP(B)} \cdot m(X).$$

$PrScT$ transformation satisfies our assumption, thus it is really *ulb*-consistent.

Sudano's *hybrid pignistic probability transformation* $PrHybT$ [37] is also *ulb*-consistent.

$$PrHybP(A) = \sum_{A \in X} \frac{PraPlP(A)}{\sum_{B \in X} PraPlP(B)} \cdot m(X).$$

[12] A counter-example: $m(\{a\}) = m(\{b\}) = m(\{c\}) = 0.1$, $m(\{a, b\}) = 0.7$, we obtain $PrPl(\{a\}) = PrPl(\{b\}) = 0.4294$ and $PrPl(\{c\}) = 0.1412 > 0.1 = Pl(\{c\})$.

Analogically to starting a proportional transformation from the bbms or the beliefs of singletons $m(a) = bel(A)$ and adding some proportions of $m(X)$ to it for $A \in X$, we can start from $Pl(A)$ and remove some proportions of $m(X)$ from it, see [13]. In this way transformations $Prop_{-Pl}T$ and $Prop_{-Bel}T$ were defined in [13]. Unfortunately their definitions in [13] are not correct in general, thus we omit them in the present text. The transformations work for 2D belief functions, but from nD quasi-Bayesian BFs problems start.

4 Summary of consistencies of probabilistic transformations

The reason of defining the new transformations in [13] was an endeavour to find a probabilistic transformation which is both $\oplus$-consistent and *ulb*-consistent or $\circledcirc$-consistent and *ulb*-consistent. This endeavour was unsuccessful, on contrary it is possible to prove the following theorem.

Theorem 2 *(i) Pl_T is the only $\oplus$-consistent probabilistic transformation which is also p-consistent.*
(ii) Bel_T is the only $\circledcirc$-consistent PT which is also p-consistent.
(iii) $BetT$ is the only α-consistent PT which is also p-consistent and satisfies Smets' assumptions of Anonymity and of Impossible event, see Section 5 and [35].

Proof: (i) Let us show that for any $\oplus$-consistent PT T and any belief function X it holds that $T(X) = Pl\_T(X)$: $T(X) = T(X) \oplus 0' = T(X) \oplus T(0') = T(X \oplus 0') = T(Pl\_T(X)) = Pl\_T(X)$. We use that $T(X)$ is Bayesian BF, that $0'$ is neutral for Bayesian BFs, that $T(0') = 0'$, $\oplus$-consitency of T, and p-consitency of T.
(ii) In an analogous way, we show that for any $\circledcirc$-consistent and p-consistent PT T and any belief function X it holds that $T(X) = Bel\_T(X)$: $T(X) = T(X)\circledcirc 0' = T(X)\circledcirc T(0') = u(T(X)\circledcirc T(0')) = T(X)(\circledcirc \circ u)T(0') = T(X(\circledcirc \circ u)0') = T(u(X\circledcirc 0')) = u(X\circledcirc 0') = Bel\_T(X\circledcirc 0') = u(Bel\_T(X\circledcirc 0')) = Bel\_T(X(\circledcirc \circ u)0') = Bel\_T(X)$. Here we use in addtition that $\circledcirc \circ u$ coincides with $\oplus$ (and Bayes' rule of probability combination, see [14]) on Bayesian BFs.
(iii) follows Smets' necessity of pignistic transformation [35].

From Theorem 2 the following corollary immediately follows.

Corollary 3 *(i) There does not exist any probabilistic transformation which is both $\oplus$-consistent and ulb-consistent in full generality. The only exception is normalized plausibility transformation Pl_T on the domain of quasi-Bayesian belief functions.*
(ii) There does not exist any probabilistic transformation which is both $\circledcirc$-consistent and ulb-consistent in full generality. The only exception is normalized belief transformation Bel_T on the domain of quasi-Bayesian belief functions.
(iii) There does not exist any $\oplus$- or $\circledcirc$-consistent probabilistic transformation which satisfies Smets' assumptions of pignistic transformation.
(iv) The pignistic transformation is neither compatible with the Dempster's rule $\oplus$ nor with the disjunctive rule of combination $\circledcirc$. (We mean compatibility in the sense of combination of pignistic transformations).

Hence there is no need to look for another new probabilistic transformation.

We can summarize consistencies of probabilistic transformations in Table 4.1.

We have to recall the following equivalencies: $Pl\_T \equiv CautT \equiv Q\_T \equiv VBT \equiv PrNPlT$, $Bel\_T \equiv DisjT$, $Prop_{Bel}T \equiv PrBlT$, and $Prop_{Pl}T \equiv Prop_{Q}T \equiv PrPlT$. On 2D BFs and on nD quasi-Bayesian BFs (qBBFs) it holds further $Bel\_T \equiv Prop_{Bel}T \equiv StBel\_T$, and $Prop_{Pl}T \equiv PrPlT \equiv PraPlT$. The equivalency $\oplus \equiv \circledcirc \circ u \equiv \otimes$ holds on general nD Bayesian BFs, see [14].

5 Justification of probabilistic transformations

The recent justification of pignistic transformation is presented in [34, 35]. Let us make a general justification of the probabilistic transformations, which have been studied in this text.

Table 4.1: Consistencies of probabilistic tranformations.

	$\oplus$-consistency	$\oplus_I$-consistency	$\circledcirc$-consistency	$\circledcirc_I$-consistency	α-consistency	ulb-consistency	p-consistency
Pl_T	$\oplus$-consistent	$\oplus_I$-consist.	no	$\circledcirc_I$-consist.	no	2D BFs nD qBBFs	Yes
Bel_T^*	no	$\oplus_I$-consist.**	$\circledcirc_I$-consistent	$\circledcirc_I$-consist.**	no	2D BFs nD qBBFs	Yes
$BetT$	no	2D BFs nD qBBFs	no	$\circledcirc_I$-consist.	α-consistent	ulb-consist.	Yes
$Prop_{Bel}T$	no	2D BFs nD qBBFs	2D BFs - $(0,0)$ nD qBBFs - ${}_{nD}0$	$\circledcirc_I$-consist.	no	ulb-consistent	Yes
$StBel\_T$	no	2D BFs nD qBBFs	2D BFs - $(0,0)$ nD qBBFs - ${}_{nD}0$	$\circledcirc_I$-consist.	no	ulb-consist.	Yes
$Prop_{Pl}T$	no	2D BFs nD qBBFs	no	$\circledcirc_I$-consist.	no	ulb-consistent	Yes
$PraPlT$	no	2D BFs nD qBBFs	no	$\circledcirc_I$-consist.	no	2D BFs nD qBBFs	Yes

* Bel_T is not defined for BFs such that $\sum_{A\in\Omega} m(A) = 0$.
** Only on definition domain of Bel_T.
qBBFs stands for quasi-Bayesian belief functions.
All the presented transformations are p-consistent,
thus all these transformations are also $\oplus$-, $\circledcirc$-, and α-consistent on nD Bayesian BFs.

Let us assume that a general probabilistic transformation PT is a function from the set of all belief functions to the Bayesian ones, i. e. to the set of probabilistic distributions on Ω. $PT(m) = P$, where $P(X) = PT(m)(X) = m'(X)$. It includes Smets' assumption of *Credal-Pignistic Link*, see Proposition 3.1 in [35]. Smets' assumption of *Efficiency*, see Proposition 4.1 in [35], also holds because $P(\Omega) = \sum_{A\in\Omega} P(A) = \sum_{A\in\Omega} m'(A) = bel'(\Omega) = 1$. All the studied transformations are p-consistent, thus we can, without lost of generality, assume this very natural assumption which requires that Bayesian BFs are transformed back to themselves. It corresponds to the Smets' *Projectivity* assumption, see Proposition 3.2 from [35].

All our probabilistic transformations satisfy also the Smets' assumption of *Anonymity*, i.e. independence of the result of transformation on permutation of elements of Ω, see Proposition 4.2 in [35], and the assumption of *Impossible event* requiring probability of an impossible event equal to zero, see Proposition 4.3 in [35].

The *Linearity assumption*, see Proposition 1.1 in [35], i.e. α-consistency in our terminology, is the only Smets' assumption that we do not include in our general assumptions. We can summarize our assumptions to the following definition.

Definition 4 *A function PT from the set of all belief functions to the set of the Bayesian ones is called* probabilistic transformation *of belief functions if it satisfies:*
(i) p-consistency, i. e. $PT(bel) = bel$ for any Bayesian BF bel,
(ii) $PT(bel)(X) = 0$ for any impossible event X, i.e. for X such that $Pl(X) = 0$,
(iii) anonymity, i.e. $TP(bel^*)(R(X)) = P^*(R(X)) = P(X) = TP(bel)(X)$, for any permutation R of elements of Ω and BF bel^* given by $m^*(R(X)) = m(X)$.

Theorem 5 *Let us assume all the assumptions from Definition 1. The following holds:*
(i) If we add an assumption (iv-a) of α-consistency, we obtain a justification of the pignistic transformation $BetT$.
(ii) If we add an assumption (iv-c) of $\oplus$-consistency, we obtain a justification of the normalized plausibility transformation Pl_T.
(iii) If we add an assumption (iv-d) of $\circledcirc$-consistency, we obtain a justification of the normalized belief transformation Bel_T.

The proofs of the statements immediately follow Definition 1, Theorem 2, and properties of the

transformations. Note that both Cobb & Shenoy's *Invariance with respect to combination* and *Idempotency* [3] follow the assumption (iv-c) of $\oplus$-consistency.

The addition of an assumption of the *ulb*-consistency does not justify any unique probabilistic transformation. On the other hand, it excludes Pl_T and Bel_T, hence we do not assume any *ulb*-consistency in our new definition of probabilistic transformations.

6 Applicability of probabilistic transformations

Several probabilistic transformations have been presented and compared in this text. None of them is the best of all in general. Thus a natural question arises: Which probabilistic transformation should be used in our applications? As the answer is not unique, we will discuss it in this section.

The answer depends on the reason why we want to compute the probabilistic transformation and how we want to use it: Whether our goal is only to find the most prospective element of the frame of discernment or whether we have some specific assumptions to the result, and what operations we want to perform with the resulting probability.

Let us assume that we have all our evidence represented with BFs, i.e. that there is no other explicit nor implicit information about bbms assigned to multi-element focal elements. If we want to use a transformed probability for betting, we have to follow the Smets' necessity of pignistic transformation and compute pignistic probabilities. Nevertheless, we have to use them strictly on the pignistic level and to keep in mind that we cannot handle pignistic probabilities like the Bayesian BFs and combine them with the conjunctive or disjunctive rule of combination and similarly.

If we assume that the belief corresponds to lower probability and the plausibility to upper probability, we have to use some of the *ulb*-consistent probabilistic transformations. Similarly as before, we have to keep in mind that we have left the credal level and that we cannot handle probabilities as Bayesian BFs. If we, moreover, assume the α-consistency, then it is the only possibility of the pignistic probability again.

If we assume or want to be prepared for a combination of the resulting probabilities with the conjunctive combination, we have to use $\oplus$-consistent transformation, i.e. Pl_T. It is just the case of Cobb & Shenoy's assumptions. Similarly, if we assume disjunctive or α-combination of the resulting probabilities we have to use $\odot$- or α-consistent transformation, i.e Bel_T or $BetT$ respectively.

If we are interested in selection of the most plausible element we have to use normalized plausibility transformation Pl_T. For determining the most believable element we have to use normalized belief Bel_T or preferably its stepwise version $StBel\_T$. In the case where $\odot$ rule and Bel_T are used, we can handle probability as a Bayesian belief and combine it with $\odot$. While in the case $StBel\_T$ we have to keep in mind that the credal level was left.

In the case of general looking for the most prospective element of the frame of discernment (without any other assumption) we can select a transformation with regard to its interpretation, see [12, 15].

If we have some other information on the domain, on the belief functions which are transformed or some special requirements to the resulting probabilities, we can use some special probabilistic transformation.

We assume that the evidence about application domain is represented with belief functions. It is called the *credal level* by Smets. By applying the pignistic transformation we leave this level and move us to the *pignistic level*. In the case that we do not assume α-consistency and do not use the pignistic transformations, we cannot speak longer about the pignistic level than about the *probabilistic level* or, more generally, about the *decisional level* of a representation and a solution of the decisional task.

7 Conclusion

A series of probabilistic transformations of belief functions have been analyzed and compared in this text, namely from the point of view of combination consistencies. They have different pros and cons. It has been shown that there does not exist a probabilistic transformation which is the best in general.

A new definition of probabilistic transformations which covers all the investigated transformations has been presented.

A particular discussion about which transformation should be applied in applications concludes the study. It has been shown that both the Smets' approach of the necessity of the pignistic transformation and the Cobb & Shenoy's necessity of the normalized plausibility transformation are right within their assumptions which are mutually different. Besides, the other assumptions tend to other alternative solutions.

Bibliography

[1] Appriou, A.: Probabilités et Incertitude en Fusion de Doneés Multisenseurs. Revue Scientifique et Technique de la Défense **11** (1991) 27–40.

[2] Baroni, P., Vicig, P.: Transformations from Imprecise to Precise Probabilities. In Nielsen, T. D., Zhang, N. L., editors, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2003); Lecture Notes in Artificial Intelligence 2711* , Springer-Verlag (2003) 37–49.

[3] Cobb, B. R., Shenoy, P. P.: A Comparison of Methods for Transforming Belief Functions Models to Probability Models. In Nielsen, T. D., Zhang, N. L., editors, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2003); Lecture Notes in Artificial Intelligence 2711* , Springer-Verlag (2003) 255–266.

[4] Cohen, M.S.: An expert system framework for non-monotonic reasoning about probabilistic assumptions, In Kanal, L.N., Lemmer, J.F. (eds.): *Uncertainty in Artificial Intelligence 1.* North-Holland (1986).

[5] Daniel, M.: Distribution of Contradictive Belief Masses in Combination of Belief Functions. In: Bouchon-Meunier, B., Yager, R. R., Zadeh, L. A. (eds.): *Information, Uncertainty and Fusion.* Kluwer Academic Publishers (2000) 431–446.

[6] Daniel, M.: Associativity and Contradiction in Combination of Belief Functions. In: *Proceedings Eight International Conference Information Processing and Management of Uncertainty, IPMU* (2000) 133–140.

[7] Daniel, M.: Combination of Belief Functions and Coarsening/Refinement. In *Proceedings Ninth International conference IPMU*, Université de Savoie, Annecy, Vol. I. (2002) 587–594.

[8] Daniel, M.: Associativity in combination of belief functions; a derivation of minC combination. *Soft Computing* **7** (2003) 288–296.

[9] Daniel, M.: Algebraic Structures Related to the Combination of Belief Functions. *Scientiae Mathematicae Japonicae* **60**/ 2 (2004) 245–255. *Scientiae Mathematicae Japonicae Online* **10** (2004) 501–511.

[10] Daniel, M.: Algebraic Structures Related to the Consensus Operator for Combining of Beliefs. In Nielsen, T. D., Zhang, N. L., editors, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2003); LNAI 2711*, Springer-Verlag (2003) 332–344.

[11] Daniel., M.: Combination of Belief Functions on Two-element and on General n-element Frame of Discernment In Ramík, J., editor, *Proceedings of 6th Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty* (2003).

[12] Daniel., M.: Transformations of Belief Functions to Probabilities. In Vejnarová, J., editor, *Proceedings of 6th Workshop on Uncertainty Processing (WUPES 2003); VŠE - Oeconomica Publishers, 2003, pp. 77–90.*

[13] Daniel, M.: Consistency of Probabilistic Transformations of Belief Functions. In Proceedings of the Tenth International conference IPMU *(2004) 1135–1142 .*

[14] Daniel, M.: A Relation of Conjunctive and Disjunctive Rules of Combination on Bayesian Belief Functions. In Noguchi, H., Ishii, H., Inuiguchi, M., editors, Proceedings of 7th Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty *(2004) 179–184.*

[15] Daniel., M.: Transformations of Belief Functions to Probabilities. International Journal of Intelligent Systems. *(in print).*

[16] Dempster, A. P.: How to "sell" the Dempster-Shafer theory. Oral presentation in WUPES 2003, Hejnice, Czech republic, September 24–27, 2003.

[17] Dezert, J.: Foundations for a New Theory of Plausible and Paradoxical Reasoning. Information and Security Journal **9** *(2002).*

[18] *Dezert, J., Smarandache, F., Daniel, M.: The Generalized Pignistic Transformation. In Svensson, P., Schubert, J. (eds.), Proceedings of the Seventh International Conference om Information Fusion, FUSION 2004 (2004) 384–391.*

[19] *Dubois, D., Prade, H.: On several representations of an uncertain body of evidence. In Gupta, M. M., Sanchez, E., editors,* Fuzzy Information and Decision Processes *North-Holland, Amsterdam (1982) 167–181.*

[20] *Dubois D., Prade H. (1986), A Set-Theoretic View of Belief Functions.* Int. J. General Systems, **12** *193–226.*

[21] *Dubois D., Prade H. (1988), Representation an combination of uncertainty with belief functions and possibility measures.* Computational Intelligence, **4** *244–264.*

[22] *Dubois, D., Prade, H.: Consonant Approximations of Belief Functions.* International Journal of Approximate Reasoning **4** *(1990) 419–449.*

[23] *Haenni R. (2005), Shedding New Light on Zadeh's Criticism of Dempster's Rule of Combination. In:* Proceedings of Information Fusion 2005, *Philadelphia, July 2005.*

[24] *Hájek, P., Hájková, M., Havránek, T., Daniel, M.: The Expert System Shell EQUANT-PC: Brief information. Kybernetika 1–3* **25** *(1989) suppl. 4–9.*

[25] *Hájek, P., Havránek, T., Jiroušek, R.:* Uncertain Information Processing in Expert Systems. *CRC Press, Boca Raton, Florida (1992).*

[26] *Hájek, P., Valdés, J. J.: Generalized algebraic foundations of uncertainty processing in rule-based expert systems (dempsteroids).* Computers and Artificial Intelligence **10** *(1991) 29–42.*

[27] *Chateauneuf, A., Jaffray, J.-Y.: Some Characterizations of lower probabilities and other monotone capacities through the use of Moebius Inversion.* Mathematical Social Sciences **17** *(1989) 263-283.*

[28] *Jøsang, A.: The Consensus Operator for Combining Beliefs.* Artificial Intelligence Journal **141/1–2** *(2002) 157–170.*

[29] *Lefevre, E., Colot, O., Vannoorenberghe, P.: Belief Functions Combination and Conflict Management.* Information Fusion **3/2** *(2002) 149–162.*

[30] *Shafer, G.:* A Mathematical Theory of Evidence. *Princeton University Press, Princeton, New Jersey (1976).*

[31] *Smets, Ph.: Constructing the Pignistic Probability Function in a Context of Uncertainty. In Henrion, M., Schachter, R. D., Kanal, L. N., Lemmer, J. F. (eds.)* Uncertainty in Artificial Intelligence 5 *Amsterdam, North Holland (1990) 29–39.*

[32] *Smets, Ph.: The combination of evidence in the transferable belief model.* IEEE-Pattern analysis and Machine Intelligence **12** *(1990) 447–458*

[33] *Smets, Ph., Kennes, R.: The transferable belief model.* Artificial Intelligence **66** *(1994) 191–234.*

[34] *Smets, Ph.: Decision Making in a Context where Uncertainty is Represented by Belief Functions. In Srivastava, R. P., Mock, T. J. (eds.)* Belief Functions in Business Decision *Physica-Verlag, Heidelberg, Germany, (2002) 17–61.*

[35] *Smets, Ph.: Decision Making in the TBM: the Necessity of the Pignistic Transformation. (*International Journal of Approximative Reasoning, *In print.)*

[36] *Sudano, J. J.: Pignistic Probability transforms for Mixes of Low- and High-Probability Events. In Proc. of the 4th Int. Conf. on Information Fusion (Fusion 2001), Montreal, Canada (2001) TUB3 23–27/9.*

[37] *Sudano, J. J.: Equivalence Between Belief Theories and Naive Bayesian Fusion for Systems with Independent Evidential Data: Part II, the Example. In Proc. of the 6th Int. Conf. on Information Fusion (Fusion 2003), Cairns, Australia, (2003) 1357–1364.*

[38] *Voorbraak, F.: A Computationally Efficient Approximation of Dempster-Shafer Theory.* International Journal of Man-Machine Studies **30** *(1989) 525–536.*

[39] *Williams, P. M.: Discussion of Shafer G. "Belief Functions and Parametric Models".* Journal of Royal Statistical Society **B44** *(1982) 342 et seq.*

[40] *Yager R. R. (1987), On the Demspter-Shafer framework and new combination rules.* Information Sciences, **41** *93–138.*