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Bounds on Eigenvalues of Interval Matrices

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Technical report No. 688

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Abstract

We describe a rectangle in the complex plane enclosing all eigenvalues of an interval matrix $A'$. We give theoretical bounds (Theorem 1) that are exact for symmetric or skew-symmetric matrices (Theorem 2) and practical bounds (Theorem 3) requiring evaluation of 6 minimal or maximal eigenvalues of symmetric matrices. Some consequences are mentioned.

Keywords

Interval matrix, eigenvalue, bound

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1 Theoretical bounds

We consider square interval matrices in the form

\[ A^I = [A_c - \Delta, A_c + \Delta] = \{ A; A_c - \Delta \leq A \leq A_c + \Delta \} \]

where inequalities are understood componentwise; thus \( A_c \) is the center matrix and \( \Delta \) is the radius matrix of \( A^I \).

**Theorem 1** Let \( A^I = [A_c - \Delta, A_c + \Delta] \) be a square interval matrix. Then for each eigenvalue \( \lambda \) of each \( A \in A^I \) we have

\[ r \leq \text{Re} \lambda \leq \overline{r}, \tag{1.1} \]

\[ \underline{i} \leq \text{Im} \lambda \leq \overline{i}, \tag{1.2} \]

where

\[ \underline{r} = \min_{\|x\|_2 = 1} (x^T A_c x - |x|^T \Delta |x|), \]

\[ \overline{r} = \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|), \]

\[ \underline{i} = \min_{\|(x_1, x_2)\|_2 = 1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|), \]

\[ \overline{i} = \max_{\|(x_1, x_2)\|_2 = 1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|). \]

**Comments.** Vectors are always considered column vectors, so that \( x^T y \) is the scalar product whereas \( xy^T \) is the matrix \((x_i y_j)\). In the formulae for \( \underline{i} \) and \( \overline{i} \), for typographic reasons we write “\( \|(x_1, x_2)\|_2 = 1 \)” in the subscript instead of the correct “\( \|(x_1^T, x_2^T)^T\|_2 = 1 \)”. For \( A = (a_{ij}) \) and \( B = (b_{ij}) \) we use

\[ A \circ B = \sum_{ij} a_{ij} b_{ij} \]

("scalar product of matrices"). Then we have

\[ x^T A y = \sum_{ij} x_i a_{ij} y_j = A \circ (xy^T). \]

**Proof.** Let \( \lambda = \lambda_1 + \lambda_2 i \) be an eigenvalue of some \( A \in A^I \). Then

\[ A(x_1 + x_2 i) = (\lambda_1 + \lambda_2 i)(x_1 + x_2 i) \]

(1.3)

for some real vectors \( x_1, x_2, x_1 \neq 0 \) or \( x_2 \neq 0 \), which may be normalized to achieve

\[ x_1^T x_1 + x_2^T x_2 = 1. \]

(1.4)

Premultiplying (1.3) by the complex conjugate vector \( x_1 - x_2 i \), we obtain

\[ \lambda_1 + \lambda_2 i = (x_1 - x_2 i)^T A(x_1 + x_2 i), \]

1
which yields
\[
\begin{align*}
\text{Re} \lambda &= \lambda_1 = x_1^T A x_1 + x_2^T A x_2, \\
\text{Im} \lambda &= \lambda_2 = x_1^T A x_2 - x_2^T A x_1. 
\end{align*}
\] (1.5)

1) To prove that \( \text{Re} \lambda \leq \overline{\tau} \), denote \( r(A) = \max_{\|x\|_2 = 1} x^T A x \), then we have
\[
\begin{align*}
x_1^T A x_1 &\leq r(A) x_1^T x_1, \\
x_2^T A x_2 &\leq r(A) x_2^T x_2,
\end{align*}
\]

hence
\[
x_1^T A x_1 + x_2^T A x_2 \leq r(A) (x_1^T x_1 + x_2^T x_2) = r(A) \] (1.7)
due to (1.4), and
\[
\begin{align*}
\begin{multlined}[t][.9\linewidth]
r(A) = \max_{\|x\|_2 = 1} x^T A x = \max_{\|x\|_2 = 1} (x^T A x + x^T (A - A_c) x) \\
\leq \max_{\|x\|_2 = 1} (x^T A x + |x|^T \Delta |x|) = \overline{\tau}.
\end{multlined}
\end{align*}
\] (1.8)

Then from (1.5), (1.7) and (1.8) we obtain
\[
\begin{align*}
\text{Re} \lambda &\leq \overline{\tau},
\end{align*}
\]
which is the right-hand side inequality in (1.1).

2) Since \( -\lambda \) is an eigenvalue of \( -A \) which belongs to \([-A_c - \Delta, -A_c + \Delta]\), from the result proved in 1) applied to \([-A_c - \Delta, -A_c + \Delta]\) we obtain
\[
-\text{Re} \lambda = \text{Re} (-\lambda) \leq \max_{\|x\|_2 = 1} (-x^T A_c x + |x|^T \Delta |x|),
\]
which implies
\[
\begin{align*}
\text{Re} \lambda &\geq - \max_{\|x\|_2 = 1} (-x^T A_c x + |x|^T \Delta |x|) = \min_{\|x\|_2 = 1} (x^T A_c x - |x|^T \Delta |x|) = \underline{\tau}, 
\end{align*}
\]
which is the left-hand side inequality in (1.1).

3) Since
\[
\begin{align*}
x_1^T A x_2 - x_2^T A x_1 &= x_1^T A x_2 - x_2^T A x_1 + x_1^T (A - A_c) x_2 - x_2^T (A - A_c) x_1 \\
&= x_1^T (A - A_c^T) x_2 + (A - A_c) \circ (x_1 x_2^T - x_2 x_1^T) \\
&\leq x_1^T (A - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|,
\end{align*}
\]
from (1.6) and (1.4) we get
\[
\begin{align*}
\text{Im} \lambda &\leq \max_{\| (x_1, x_2) \|_2 = 1} (x_1^T (A - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \overline{\tau},
\end{align*}
\]
which is the right-hand side inequality in (1.2).

4) Since \( -\lambda \) is an eigenvalue of \( -A \in [-A_c - \Delta, -A_c + \Delta] \), applying the result in 3) we obtain
\[
\begin{align*}
-\text{Im} \lambda = \text{Im} (-\lambda) &\leq \max_{\| (x_1, x_2) \|_2 = 1} (x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|),
\end{align*}
\]
and thereby also
\[
\begin{align*}
\text{Im} \lambda &\geq \min_{\| (x_1, x_2) \|_2 = 1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \underline{\tau},
\end{align*}
\]
which concludes the proof. \( \blacksquare \)
2 The bounds are exact in special cases

A real matrix $A$ is called symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$. An interval matrix $A^I$ is said to be symmetric if

$$A^I A^T = A^I$$

and skew-symmetric if

$$A^I A^T = -A^I,$$

where

$$A^I A^T = \{A^T; A \in A^I\}$$

and

$$-A^I = \{-A; A \in A^I\}.$$

Hence, $A^I = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if $[A_c^T - \Delta^T, A_c^T + \Delta^T] = [A_c - \Delta, A_c + \Delta]$, which is equivalent to symmetry of both $A_c$ and $\Delta$. Similarly, $A^I$ is skew-symmetric if and only if $A_c$ is skew-symmetric and $\Delta$ is symmetric.

**Theorem 2** The bounds (1.1) are exact (i.e., achieved over $A^I$) if $A^I$ is symmetric and the bounds (1.2) are exact if $A^I$ is skew-symmetric.

**Proof.** 1) Let $A^I$ be symmetric, so that $A_c$ and $\Delta$ are symmetric. Since the continuous mapping $x \mapsto x^T A_c x + |x|^T \Delta |x|$ achieves its maximum over the unit sphere $\{x; \|x\|_2 = 1\}$, there exists an $x$ satisfying

$$\tau = x^T A_c x + |x|^T \Delta |x|$$

(2.1)

and $\|x\|_2 = 1$. Define a diagonal matrix $S$ by

$$S_{jj} = \begin{cases} 1 & \text{if } x_j \geq 0, \\ -1 & \text{if } x_j < 0 \end{cases}$$

($j = 1, \ldots, n$), then $|x| = S x$ and from (2.1) we have

$$\tau = x^T A_c x + x^T S \Delta S x = x^T (A_c + S \Delta S) x \leq \lambda_{\text{max}}(A_c + S \Delta S),$$

(2.2)

where $\lambda_{\text{max}}(A_c + S \Delta S)$ denotes the maximal eigenvalue of $A_c + S \Delta S$ (which is symmetric since both $A_c$ and $\Delta$ are symmetric). Since $[S \Delta S] = \Delta$, the matrix $A_c + S \Delta S$ belongs to $A^I$, hence

$$\lambda_{\text{max}}(A_c + S \Delta S) \leq \tau$$

by Theorem 1, which combined with (2.2) gives

$$\tau = \lambda_{\text{max}}(A_c + S \Delta S),$$

hence $\tau$ is achieved over $A^I$ (even more, it is achieved at a symmetric matrix in $A^I$, cf. Hertz [1]). The proof for $\underline{\tau}$ is analogous; in this case we obtain

$$\underline{\tau} = \lambda_{\text{min}}(A_c - S \Delta S).$$
2) Let $A^I$ be skew-symmetric, so that $A_z$ is skew-symmetric and $\Delta$ is symmetric. We have
\[
\tau = x_1^T(A_z - A_z^T)x_2 + \Delta \circ |x_1x_2^T - x_2x_1^T| \tag{2.3}
\]
for some $x_1, x_2$ satisfying $\|(x_1, x_2)\|_2 = 1$. Define
\[
z_{ij} = \begin{cases} 
-1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j < 0, \\
0 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j = 0, \\
1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j > 0
\end{cases}
\]
for each $i, j$, then $z_{ij} = -z_{ji}$, hence the matrix $\tilde{\Delta}$ defined by
\[
\tilde{\Delta}_{ij} = z_{ij}\Delta_{ij}
\]
is skew-symmetric (since $\Delta$ is symmetric). Let
\[
A = A_z + \tilde{\Delta},
\]
then $A \in A^I$ and $A$ is skew-symmetric (since both $A_z$ and $\tilde{\Delta}$ are skew-symmetric). Next, from (2.3) we have
\[
\tau = x_1^T(A_z - A_z^T)x_2 + \sum_{ij} \Delta_{ij}z_{ij}(x_1x_2^T - x_2x_1^T)_{ij}
\]
\[
= x_1^T(A_z - A_z^T)x_2 + \Delta \circ (x_1x_2^T - x_2x_1^T)
\]
\[
= x_1^T(A_z - A_z^T)x_2 + x_1^T\tilde{\Delta}x_2 - x_2^T\tilde{\Delta}x_1
\]
\[
= x_1^TAx_2 - x_2^TAx_1
\]
\[
= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]
where the matrix
\[
\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}
\]
is symmetric since $A$ is skew-symmetric. Denote by $\lambda$ the maximal eigenvalue of (2.4) (which is real), then from the above expression for $\tau$ we have
\[
\tau \leq \lambda \tag{2.5}
\]
and there exists a vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0$ satisfying
\[
\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]
which implies
\[
A(y_1 + y_2i) = -\lambda y_2 + \lambda iy_1 = \lambda i(y_1 + y_2i),
\]
thus $\lambda i$ is an eigenvalue of $A$. Hence
\[
\lambda \leq \tau
\]
by Theorem 1, which combined with (2.5) gives
\[ \tau = \lambda = \text{Im} (\lambda i), \]
hence \( \tau \) is achieved as the imaginary part of an eigenvalue of a matrix in \( A^I \). To prove an analogous result for \( i \), let us apply the result just proved to the interval matrix \([-A_c - \Delta, -A_c + \Delta]\), which is also skew-symmetric. Then we have
\[
\text{Im} \lambda = \max_{\|x_1, x_2\|=1} (x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|)
\]
for an eigenvalue \( \lambda \) of some \( \tilde{A} \in [-A_c - \Delta, -A_c + \Delta] \), hence
\[
\text{Im}(-\lambda) = -\text{Im} \lambda = \min_{\|x_1, x_2\|=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \tilde{i}
\]
for the eigenvalue \(-\lambda \) of \(-\tilde{A} \in [A_c, -\Delta, A_c + \Delta] \), which shows that \( \tilde{i} \) is achieved as well. ■

3 Practical bounds

**Theorem 3** Let \( A^I = [A_c - \Delta, A_c + \Delta] \) be a square interval matrix. Then for each eigenvalue \( \lambda \) of each \( A \in A^I \) we have
\[
\lambda_{\text{min}}(A'_c) - \lambda_{\text{max}}(\Delta') \leq \text{Re} \lambda \leq \lambda_{\text{max}}(A'_c) + \lambda_{\text{max}}(\Delta'), \tag{3.1}
\]
\[
\lambda_{\text{min}}(A''_c) - \lambda_{\text{max}}(\Delta'') \leq \text{Im} \lambda \leq \lambda_{\text{max}}(A''_c) + \lambda_{\text{max}}(\Delta''), \tag{3.2}
\]
where
\[
A'_c = \frac{1}{2} (A_c + A_c^T),
\]
\[
\Delta' = \frac{1}{2} (\Delta + \Delta^T),
\]
\[
A''_c = \begin{pmatrix} 0 & \frac{1}{2} (A_c - A_c^T) \\ \frac{1}{2} (A_c^T - A_c) & 0 \end{pmatrix},
\]
\[
\Delta'' = \begin{pmatrix} \Delta' & 0 \\ 0 & \Delta' \end{pmatrix}.
\]

**Comments.** \( \lambda_{\text{min}}, \lambda_{\text{max}} \) denote the minimal and maximal eigenvalue of a symmetric matrix, respectively. Notice that all the matrices \( A'_c, \Delta', A''_c, \Delta'' \) are symmetric by definition. Since \( \lambda_{\text{max}}(D) = \rho(D) \) (spectral radius) holds for a nonnegative symmetric matrix \( D \), the formulae (3.1), (3.2) may also be written in the form
\[
\lambda_{\text{min}}(A'_c) - \rho(\Delta') \leq \text{Re} \lambda \leq \lambda_{\text{max}}(A'_c) + \rho(\Delta'),
\]
\[
\lambda_{\text{min}}(A''_c) - \rho(\Delta'') \leq \text{Im} \lambda \leq \lambda_{\text{max}}(A''_c) + \rho(\Delta'').
\]

**Proof.** Let \( \lambda \) be an eigenvalue of a matrix \( A \in A^I \).
1) Since
\[ \tau = \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|) \]
\[ \leq \max_{\|x\|_2 = 1} x^T A_c x + \max_{\|x\|_2 = 1} |x|^T \Delta |x| \]
\[ = \max_{\|x\|_2 = 1} x^T A'_c x + \max_{\|x\|_2 = 1} |x|^T \Delta' |x| \]
\[ = \lambda_{\text{max}}(A'_c) + \lambda_{\text{max}}(\Delta'), \]
by Theorem 1 there holds
\[ \text{Re} \lambda \leq \lambda_{\text{max}}(A'_c) + \lambda_{\text{max}}(\Delta'), \]
which is the right-hand side inequality in (3.1).

2) The proof of the left-hand side inequality is analogous since
\[ \tau \geq \min_{\|x\|_2 = 1} x^T A_c x - \max_{\|x\|_2 = 1} |x|^T \Delta |x| = \lambda_{\text{min}}(A'_c) - \lambda_{\text{max}}(\Delta'). \]

3) We have
\[ \tau = \max_{\|x_1, x_2\|_2 = 1} \left( x_1^T (A_c - A'_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T| \right) \]
\[ \leq \max_{\|x_1, x_2\|_2 = 1} \left( x_1^T A_c x_2 - x_2^T A_c x_1 \right) + \max_{\|x_1, x_2\|_2 = 1} \left( |x_1|^T \Delta |x_2| + |x_2|^T \Delta |x_1| \right) \]
\[ = \max_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^T \left( \begin{array}{cc} 0 & A_c \\ -A_c & 0 \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + \max_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} |x_1| \\ |x_2| \end{array} \right)^T \left( \begin{array}{cc} 0 & \Delta \\ \Delta & 0 \end{array} \right) \left( \begin{array}{c} |x_1| \\ |x_2| \end{array} \right) \]
\[ = \lambda_{\text{max}}(A''_c) + \lambda_{\text{max}}(\Delta''). \]
Hence Theorem 1 gives
\[ \text{Im} \lambda \leq \tau \leq \lambda_{\text{max}}(A''_c) + \lambda_{\text{max}}(\Delta''), \]
which is the right-hand side inequality in (3.2).

4) An analogous reasoning gives
\[ \zeta = \min_{\|x_1, x_2\|_2 = 1} \left( x_1^T (A_c - A'_c) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T| \right) \]
\[ \geq \min_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^T \left( \begin{array}{cc} 0 & \frac{1}{2} (A_c - A'_c) \\ \frac{1}{2} (A_c - A'_c) & 0 \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)
- \max_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} |x_1| \\ |x_2| \end{array} \right)^T \left( \begin{array}{cc} 0 & \frac{1}{2} (\Delta + \Delta^T) \\ \frac{1}{2} (\Delta + \Delta^T) & 0 \end{array} \right) \left( \begin{array}{c} |x_1| \\ |x_2| \end{array} \right) \]
\[ = \lambda_{\text{min}}(A''_c) - \lambda_{\text{max}}(\Delta''), \]
which in view of Theorem 1 implies the left-hand side inequality in (3.2).

\[ \blacksquare \]
4 Consequences

We keep the notations $A'_{c}, \Delta', A''_{c}, \Delta''$ introduced in Theorem 3.

**Corollary 1** If
\[ \lambda_{\max}(A'_{c}) + \lambda_{\max}(\Delta') < 0, \]
then $A'$ is (Hurwitz) stable.

**Proof.** Indeed, in this case Theorem 3 implies $\Re \lambda < 0$ for each eigenvalue $\lambda$ of each $A \in A'$.

We note that a symmetric interval matrix may contain nonsymmetric matrices with complex eigenvalues. Similarly, a skew-symmetric interval matrix may contain matrices with nonzero real parts.

**Corollary 2** If $A'$ is symmetric, then
\[ |\Im \lambda| \leq \lambda_{\max}(\Delta'') \]
for each eigenvalue $\lambda$ of each $A \in A'$.

**Proof.** The result follows from (3.2) since $A''_{c} = 0$ in view of symmetry of $A_{c}$, hence $\lambda_{\min}(A''_{c}) = \lambda_{\max}(A''_{c}) = 0$.

**Corollary 3** If $A'$ is skew-symmetric, then
\[ |\Re \lambda| \leq \lambda_{\max}(\Delta') \]
for each eigenvalue $\lambda$ of each $A \in A'$.

**Proof.** The assertion is a consequence of (3.1) since $A'_{c} = 0$.

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Bibliography
