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Bounds on Eigenvalues of Interval Matrices

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Abstract

We describe a rectangle in the complex plane enclosing all eigenvalues of an interval matrix \(A\). We give theoretical bounds (Theorem 1) that are exact for symmetric or skew-symmetric matrices (Theorem 2) and practical bounds (Theorem 3) requiring evaluation of 6 minimal or maximal eigenvalues of symmetric matrices. Some consequences are mentioned.

Keywords

Interval matrix, eigenvalue, bound

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1 Theoretical bounds

We consider square interval matrices in the form

\[ A^I = [A_c - \Delta, A_c + \Delta] = \{ A; A_c - \Delta \leq A \leq A_c + \Delta \} \]

where inequalities are understood componentwise; thus \( A_c \) is the center matrix and \( \Delta \) is the radius matrix of \( A^I \).

**Theorem 1** Let \( A^I = [A_c - \Delta, A_c + \Delta] \) be a square interval matrix. Then for each eigenvalue \( \lambda \) of each \( A \in A^I \) we have

\[ \underline{\lambda} \leq \text{Re} \lambda \leq \overline{\lambda}, \quad (1.1) \]

\[ \underline{i} \leq \text{Im} \lambda \leq \overline{i}, \quad (1.2) \]

where

\[ \underline{\lambda} = \min_{\|x\|_2 = 1} (x^T A_c x - |x|^T \Delta |x|), \]

\[ \overline{\lambda} = \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|), \]

\[ \underline{i} = \min_{\|x_1, x_2\|_2 = 1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|), \]

\[ \overline{i} = \max_{\|x_1, x_2\|_2 = 1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|). \]

**Comments.** Vectors are always considered column vectors, so that \( x^T y \) is the scalar product whereas \( x y^T \) is the matrix \((x; y_j)\). In the formulae for \( \underline{i} \) and \( \overline{i} \), for typographic reasons we write “\( \|(x_1, x_2)\|_2 = 1 \)” in the subscript instead of the correct “\( \|(x_1^T, x_2^T)^T\|_2 = 1 \)”.

For \( A = (a_{ij}) \) and \( B = (b_{ij}) \) we use

\[ A \circ B = \sum_{ij} a_{ij} b_{ij} \]

(“scalar product of matrices”). Then we have

\[ x^T A y = \sum_{ij} x_i a_{ij} y_j = A \circ (x y^T). \]

**Proof.** Let \( \lambda = \lambda_1 + \lambda_2 i \) be an eigenvalue of some \( A \in A^I \). Then

\[ A(x_1 + x_2 i) = (\lambda_1 + \lambda_2 i)(x_1 + x_2 i) \quad (1.3) \]

for some real vectors \( x_1, x_2, x_1 \neq 0 \) or \( x_2 \neq 0 \), which may be normalized to achieve

\[ x_1^T x_1 + x_2^T x_2 = 1. \quad (1.4) \]

Premultiplying (1.3) by the complex conjugate vector \( x_1 - x_2 i \), we obtain

\[ \lambda_1 + \lambda_2 i = (x_1 - x_2 i)^T A(x_1 + x_2 i), \]
which yields

\[ \Re \lambda = \lambda_1 = x_1^T A x_1 + x_2^T A x_2, \] (1.5)
\[ \Im \lambda = \lambda_2 = x_1^T A x_2 - x_2^T A x_1. \] (1.6)

1) To prove that \( \Re \lambda \leq \tau \), denote \( r(A) = \max_{\|x\|_2 = 1} x^T A x \), then we have

\[ x_1^T A x_1 \leq r(A) x_1^T x_1, \]
\[ x_2^T A x_2 \leq r(A) x_2^T x_2, \]

hence

\[ x_1^T A x_1 + x_2^T A x_2 \leq r(A) (x_1^T x_1 + x_2^T x_2) = r(A) \] (1.7)
due to (1.4), and

\[ r(A) = \max \left\{ x^T A x : \|x\|_2 = 1 \right\} = \max \left\{ (x^T A_c x + x^T (A - A_c) x) : \|x\|_2 = 1 \right\} \]
\[ \leq \max \left\{ (x^T A_c x + |x|^T \Delta |x|) : \|x\|_2 = 1 \right\} = \tau. \] (1.8)

Then from (1.5), (1.7) and (1.8) we obtain

\[ \Re \lambda \leq \tau, \]

which is the right-hand side inequality in (1.1).

2) Since \( -\lambda \) is an eigenvalue of \( -A \) which belongs to \([ -A_c - \Delta, -A_c + \Delta ] \), from the result proved in 1) applied to \([ -A_c - \Delta, -A_c + \Delta ] \) we obtain

\[ -\Re \lambda = \Re (-\lambda) \leq \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|), \]

which implies

\[ \Re \lambda \geq - \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta |x|) = \min_{\|x\|_2 = 1} (x^T A_c x - |x|^T \Delta |x|) = \tau, \]

which is the left-hand side inequality in (1.1).

3) Since

\[ x_1^T A x_2 - x_2^T A x_1 = x_1^T A_c x_2 - x_2^T A_c x_1 + x_1^T (A - A_c) x_2 - x_2^T (A - A_c) x_1 \]
\[ = x_1^T (A_c - A_c^T) x_2 + (A - A_c) \circ (x_1 x_2^T - x_2 x_1^T) \]
\[ \leq x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|, \]

from (1.6) and (1.4) we get

\[ \Im \lambda \leq \max_{\|x\|_2 = 1} \left( x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T| \right) = \tau, \]

which is the right-hand side inequality in (1.2).

4) Since \( -\lambda \) is an eigenvalue of \( -A \in [ -A_c - \Delta, -A_c + \Delta ] \), applying the result in 3) we obtain

\[ -\Im \lambda = \Im (-\lambda) \leq \max_{\|x\|_2 = 1} \left( x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T| \right) \]

and thereby also

\[ \Im \lambda \geq \min_{\|x\|_2 = 1} \left( x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T| \right) = \tau, \]

which concludes the proof.
2 The bounds are exact in special cases

A real matrix $A$ is called symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$. An interval matrix $A'$ is said to be symmetric if

$$A'^T = A'$$

and skew-symmetric if

$$A'^T = -A',$$

where

$$A'^T = \{ A^T; A \in A' \}$$

and

$$-A' = \{ -A; A \in A' \}.$$

Hence, $A' = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if $[A_c^T - \Delta^T, A_c^T + \Delta^T] = [A_c - \Delta, A_c + \Delta]$, which is equivalent to symmetry of both $A_c$ and $\Delta$. Similarly, $A'$ is skew-symmetric if and only if $A_c$ is skew-symmetric and $\Delta$ is symmetric.

**Theorem 2** The bounds (1.1) are exact (i.e., achieved over $A'$) if $A'$ is symmetric and the bounds (1.2) are exact if $A'$ is skew-symmetric.

**Proof.** 1) Let $A'$ be symmetric, so that $A_c$ and $\Delta$ are symmetric. Since the continuous mapping $x \mapsto x^TA_cx + |x|^T\Delta|x|$ achieves its maximum over the unit sphere $\{x; \|x\|_2 = 1\}$, there exists an $x$ satisfying

$$\tau = x^TA_cx + |x|^T\Delta|x|$$

and $\|x\|_2 = 1$. Define a diagonal matrix $S$ by

$S_{jj} = \begin{cases} 1 & \text{if } x_j \geq 0, \\ -1 & \text{if } x_j < 0 \end{cases}$

$(j = 1, \ldots, n)$, then $|x| = Sx$ and from (2.1) we have

$$\tau = x^TA_cx + x^TS\Delta Sx = x^T(A_c + S\Delta S)x \leq \lambda_{\max}(A_c + S\Delta S),$$

where $\lambda_{\max}(A_c + S\Delta S)$ denotes the maximal eigenvalue of $A_c + S\Delta S$ (which is symmetric since both $A_c$ and $\Delta$ are symmetric). Since $|S\Delta S| = \Delta$, the matrix $A_c + S\Delta S$ belongs to $A'$, hence

$$\lambda_{\max}(A_c + S\Delta S) \leq \tau$$

by Theorem 1, which combined with (2.2) gives

$$\tau = \lambda_{\max}(A_c + S\Delta S),$$

hence $\tau$ is achieved over $A'$ (even more, it is achieved at a symmetric matrix in $A'$, cf. Hertz [1]). The proof for $\underline{\tau}$ is analogous; in this case we obtain

$$\underline{\tau} = \lambda_{\min}(A_c - S\Delta S).$$
2) Let $A'$ be skew-symmetric, so that $A_c$ is skew-symmetric and $\Delta$ is symmetric. We have
\begin{equation}
\tau = x_1^T(A_c - A_c^T)x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T| \tag{2.3}
\end{equation}
for some $x_1, x_2$ satisfying $\|(x_1, x_2)\|_2 = 1$. Define
\[
 z_{ij} = \begin{cases} 
 -1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j < 0, \\
 0 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j = 0, \\
 1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j > 0
\end{cases}
\]
$(i, j = 1, \ldots, n)$, then $z_{ij} = -z_{ji}$ for each $i, j$, hence the matrix $\tilde{\Delta}$ defined by
\[
\tilde{\Delta}_{ij} = z_{ij}\Delta_{ij}
\]
$(i, j = 1, \ldots, n)$ is skew-symmetric (since $\Delta$ is symmetric). Let
\[
 A = A_c + \tilde{\Delta},
\]
then $A \in A'$ and $A$ is skew-symmetric (since both $A_c$ and $\tilde{\Delta}$ are skew-symmetric). Next, from (2.3) we have
\[
\tau = x_1^T(A_c - A_c^T)x_2 + \sum_{ij} \Delta_{ij}z_{ij}(x_1 x_2^T - x_2 x_1^T)_{ij}
\]
\[
= x_1^T(A_c - A_c^T)x_2 + \tilde{\Delta} \circ (x_1 x_2^T - x_2 x_1^T)
\]
\[
= x_1^T(A_c - A_c^T)x_2 + x_1^T \tilde{\Delta} x_2 - x_2^T \tilde{\Delta} x_1
\]
\[
= x_1^T A x_2 - x_2^T A x_1 + x_1^T \tilde{\Delta} x_2 - x_2^T \tilde{\Delta} x_1
\]
\[
= \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]
where the matrix
\[
\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}
\]
is symmetric since $A$ is skew-symmetric. Denote by $\lambda$ the maximal eigenvalue of (2.4) (which is real), then from the above expression for $\tau$ we have
\begin{equation}
\tau \leq \lambda \tag{2.5}
\end{equation}
and there exists a vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0$ satisfying
\[
\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]
which implies
\[
A(y_1 + y_2i) = -\lambda y_2 + \lambda iy_1 = \lambda i(y_1 + y_2i),
\]
thus $\lambda i$ is an eigenvalue of $A$. Hence
\[
\lambda \leq \tau
\]
by Theorem 1, which combined with (2.5) gives
\[ \tau = \lambda = \text{Im}(\lambda i), \]
hence \( \tau \) is achieved as the imaginary part of an eigenvalue of a matrix in \( A^I \). To prove an analogous result for \( i \), let us apply the result just proved to the interval matrix \([-A_c - \Delta, -A_c + \Delta]\), which is also skew-symmetric. Then we have
\[ \text{Im} \lambda = \max_{\|x_1, x_2\|_2 = 1} (x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) \]
for an eigenvalue \( \lambda \) of some \( \bar{A} \in [-A_c - \Delta, A_c + \Delta] \), hence
\[ \text{Im}(-\lambda) = -\text{Im} \lambda = \min_{\|x_1, x_2\|_2 = 1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = i \]
for the eigenvalue \( -\lambda \) of \( -\bar{A} \in [A_c - \Delta, A_c + \Delta] \), which shows that \( i \) is achieved as well. \( \blacksquare \)

3 Practical bounds

**Theorem 3** Let \( A^I = [A_c - \Delta, A_c + \Delta] \) be a square interval matrix. Then for each eigenvalue \( \lambda \) of each \( A \in A^I \) we have
\[ \lambda_{\min}(A_c') - \lambda_{\max}(A_c') \leq \text{Re} \lambda \leq \lambda_{\max}(A_c') + \lambda_{\max}(\Delta'), \tag{3.1} \]
\[ \lambda_{\min}(A_c^0) - \lambda_{\max}(\Delta^0) \leq \text{Im} \lambda \leq \lambda_{\max}(A_c^0) + \lambda_{\max}(\Delta^0), \tag{3.2} \]
where
\[ A_c' = \frac{1}{2} (A_c + A_c^T), \]
\[ \Delta' = \frac{1}{2} (\Delta + \Delta^T), \]
\[ A_c^0 = \begin{pmatrix} 0 & \frac{1}{2} (A_c - A_c^T) \\ \frac{1}{2} (A_c^T - A_c) & 0 \end{pmatrix}, \]
\[ \Delta^0 = \begin{pmatrix} 0 & \Delta' \\ \Delta' & 0 \end{pmatrix}. \]

**Comments.** \( \lambda_{\min}, \lambda_{\max} \) denote the minimal and maximal eigenvalue of a symmetric matrix, respectively. Notice that all the matrices \( A_c', \Delta', A_c^0, \Delta^0 \) are symmetric by definition. Since \( \lambda_{\max}(D) = \rho(D) \) (spectral radius) holds for a nonnegative symmetric matrix \( D \), the formulae (3.1), (3.2) may also be written in the form
\[ \lambda_{\min}(A_c') - \rho(\Delta') \leq \text{Re} \lambda \leq \lambda_{\max}(A_c') + \rho(\Delta'), \]
\[ \lambda_{\min}(A_c^0) - \rho(\Delta^0) \leq \text{Im} \lambda \leq \lambda_{\max}(A_c^0) + \rho(\Delta^0). \]

**Proof.** Let \( \lambda \) be an eigenvalue of a matrix \( A \in A^I \).
1) Since
\[
\tau = \max_{\|x\|_2 = 1} (x^T A_c x + |x|^T \Delta|x|) \\
\leq \max_{\|x\|_2 = 1} x^T A_c x + \max_{\|x\|_2 = 1} |x|^T \Delta|x| \\
= \max_{\|x\|_2 = 1} x^T A'_c x + \max_{\|x\|_2 = 1} |x|^T \Delta'|x| \\
= \lambda_{\max}(A'_c) + \lambda_{\max}(\Delta'),
\]
by Theorem 1 there holds
\[
\text{Re} \lambda \leq \lambda_{\max}(A'_c) + \lambda_{\max}(\Delta'),
\]
which is the right-hand side inequality in (3.1).

2) The proof of the left-hand side inequality is analogous since
\[
\tau \geq \min_{\|x\|_2 = 1} x^T A_c x - \max_{\|x\|_2 = 1} |x|^T \Delta|x| = \lambda_{\min}(A'_c) - \lambda_{\max}(\Delta').
\]

3) We have
\[
\tau = \max_{\|x_1, x_2\|_2 = 1} (x_1^T (A_c - A'_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) \\
\leq \max_{\|x_1, x_2\|_2 = 1} (x_1^T A_c x_2 - x_2^T A_c x_1) + \max_{\|x_1, x_2\|_2 = 1} (|x_1|^T \Delta|x_2| + |x_2|^T \Delta|x_1|) \\
= \max_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c|c} x_1 \\ \hline x_2 \end{array} \right)^T \left( \begin{array}{cc} 0 & A_c \\ \hline -A_c & 0 \end{array} \right) \left( \begin{array}{c} x_1 \\ \hline x_2 \end{array} \right) + \max_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} |x_1| \\ \hline |x_2| \end{array} \right)^T \left( \begin{array}{cc} 0 & \Delta \\ \hline -\Delta & 0 \end{array} \right) \left( \begin{array}{c} |x_1| \\ \hline |x_2| \end{array} \right) \\
= \lambda_{\max}(A''_c) + \lambda_{\max}(\Delta'').
\]
Hence Theorem 1 gives
\[
\text{Im} \lambda \leq \tau \leq \lambda_{\max}(A''_c) + \lambda_{\max}(\Delta''),
\]
which is the right-hand side inequality in (3.2).

4) An analogous reasoning gives
\[
\tau = \min_{\|x_1, x_2\|_2 = 1} (x_1^T (A_c - A'_c) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) \\
\geq \min_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} x_1 \\ \hline x_2 \end{array} \right)^T \left( \begin{array}{cc} 0 & \frac{1}{2}(A_c - A'_c) \\ \hline \frac{1}{2}(A'_c - A_c) & 0 \end{array} \right) \left( \begin{array}{c} x_1 \\ \hline x_2 \end{array} \right) \\
- \max_{\|x_1, x_2\|_2 = 1} \left( \begin{array}{c} |x_1| \\ \hline |x_2| \end{array} \right)^T \left( \begin{array}{cc} 0 & \frac{1}{2}(\Delta + \Delta^T) \\ \hline \frac{1}{2}(\Delta + \Delta^T) & 0 \end{array} \right) \left( \begin{array}{c} |x_1| \\ \hline |x_2| \end{array} \right) \\
= \lambda_{\min}(A''_c) - \lambda_{\max}(\Delta''),
\]
which in view of Theorem 1 implies the left-hand side inequality in (3.2).
4 Consequences

We keep the notations $A', \Delta', A''', \Delta''$ introduced in Theorem 3.

**Corollary 1** If
\[ \lambda_{\text{max}}(A') + \lambda_{\text{max}}(\Delta') < 0, \]
then $A'$ is (Hurwitz) stable.

**Proof.** Indeed, in this case Theorem 3 implies $\text{Re} \lambda < 0$ for each eigenvalue $\lambda$ of each $A \in A'$. ■

We note that a symmetric interval matrix may contain nonsymmetric matrices with complex eigenvalues. Similarly, a skew-symmetric interval matrix may contain matrices with nonzero real parts.

**Corollary 2** If $A'$ is symmetric, then
\[ |\text{Im} \lambda| \leq \lambda_{\text{max}}(\Delta''') \]
for each eigenvalue $\lambda$ of each $A \in A'$.

**Proof.** The result follows from (3.2) since $A''' = 0$ in view of symmetry of $A$, hence $\lambda_{\text{min}}(A''') = \lambda_{\text{max}}(A''') = 0$. ■

**Corollary 3** If $A'$ is skew-symmetric, then
\[ |\text{Re} \lambda| \leq \lambda_{\text{max}}(\Delta') \]
for each eigenvalue $\lambda$ of each $A \in A'$.

**Proof.** The assertion is a consequence of (3.1) since $A' = 0$. ■

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Bibliography
