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Abstract

We describe a rectangle in the complex plane enclosing all eigenvalues of an interval matrix A^I. We give theoretical bounds (Theorem 1) that are exact for symmetric or skew-symmetric matrices (Theorem 2) and practical bounds (Theorem 3) requiring evaluation of 6 minimal or maximal eigenvalues of symmetric matrices. Some consequences are mentioned.

Keywords
Interval matrix, eigenvalue, bound

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1 Theoretical bounds

We consider square interval matrices in the form

$$A^I = [A_c - \Delta, A_c + \Delta] = \{A;\ A_c - \Delta \le A \le A_c + \Delta\}$$

where inequalities are understood componentwise; thus A_c is the center matrix and Δ is the radius matrix of A^I.

Theorem 1 *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a square interval matrix. Then for each eigenvalue λ of each $A \in A^I$ we have*

$$\underline{r} \le \operatorname{Re} \lambda \le \overline{r}, \tag{1.1}$$

$$\underline{i} \le \operatorname{Im} \lambda \le \overline{\imath}, \tag{1.2}$$

where

$$\begin{aligned}
\underline{r} &= \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|), \\
\overline{r} &= \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|), \\
\underline{i} &= \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|), \\
\overline{\imath} &= \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|).
\end{aligned}$$

Comments. Vectors are always considered column vectors, so that $x^T y$ is the scalar product whereas xy^T is the matrix $(x_i y_j)$. In the formulae for $\underline{i}$ and $\overline{\imath}$, for typographic reasons we write "$\|(x_1, x_2)\|_2 = 1$" in the subscript instead of the correct "$\|(x_1^T, x_2^T)^T\|_2 = 1$". For $A = (a_{ij})$ and $B = (b_{ij})$ we use

$$A \circ B = \sum_{ij} a_{ij} b_{ij}$$

("scalar product of matrices"). Then we have

$$x^T A y = \sum_{ij} x_i a_{ij} y_j = A \circ (xy^T).$$

Proof. Let $\lambda = \lambda_1 + \lambda_2 i$ be an eigenvalue of some $A \in A^I$. Then

$$A(x_1 + x_2 i) = (\lambda_1 + \lambda_2 i)(x_1 + x_2 i) \tag{1.3}$$

for some real vectors x_1, x_2, $x_1 \ne 0$ or $x_2 \ne 0$, which may be normalized to achieve

$$x_1^T x_1 + x_2^T x_2 = 1. \tag{1.4}$$

Premultiplying (1.3) by the complex conjugate vector $x_1 - x_2 i$, we obtain

$$\lambda_1 + \lambda_2 i = (x_1 - x_2 i)^T A(x_1 + x_2 i),$$

which yields

$$\mathrm{Re}\,\lambda = \lambda_1 = x_1^T A x_1 + x_2^T A x_2, \tag{1.5}$$

$$\mathrm{Im}\,\lambda = \lambda_2 = x_1^T A x_2 - x_2^T A x_1. \tag{1.6}$$

1) To prove that $\mathrm{Re}\,\lambda \le \overline{r}$, denote $r(A) = \max_{\|x\|_2=1} x^T A x$, then we have

$$x_1^T A x_1 \le r(A) x_1^T x_1,$$
$$x_2^T A x_2 \le r(A) x_2^T x_2,$$

hence

$$x_1^T A x_1 + x_2^T A x_2 \le r(A)(x_1^T x_1 + x_2^T x_2) = r(A) \tag{1.7}$$

due to (1.4), and

$$\begin{aligned} r(A) &= \max_{\|x\|_2=1} x^T A x = \max_{\|x\|_2=1} (x^T A_c x + x^T (A - A_c) x) \\ &\le \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|) = \overline{r}. \end{aligned} \tag{1.8}$$

Then from (1.5), (1.7) and (1.8) we obtain

$$\mathrm{Re}\,\lambda \le \overline{r},$$

which is the right-hand side inequality in (1.1).

2) Since $-\lambda$ is an eigenvalue of $-A$ which belongs to $[-A_c - \Delta, -A_c + \Delta]$, from the result proved in 1) applied to $[-A_c - \Delta, -A_c + \Delta]$ we obtain

$$-\mathrm{Re}\,\lambda = \mathrm{Re}\,(-\lambda) \le \max_{\|x\|_2=1} (-x^T A_c x + |x|^T \Delta |x|),$$

which implies

$$\mathrm{Re}\,\lambda \ge -\max_{\|x\|_2=1} (-x^T A_c x + |x|^T \Delta |x|) = \min_{\|x\|_2=1} (x^T A_c x - |x|^T \Delta |x|) = \underline{r},$$

which is the left-hand side inequality in (1.1).

3) Since

$$\begin{aligned} x_1^T A x_2 - x_2^T A x_1 &= x_1^T A_c x_2 - x_2^T A_c x_1 + x_1^T (A - A_c) x_2 - x_2^T (A - A_c) x_1 \\ &= x_1^T (A_c - A_c^T) x_2 + (A - A_c) \circ (x_1 x_2^T - x_2 x_1^T) \\ &\le x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|, \end{aligned}$$

from (1.6) and (1.4) we get

$$\mathrm{Im}\,\lambda \le \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \overline{\imath},$$

which is the right-hand side inequality in (1.2).

4) Since $-\lambda$ is an eigenvalue of $-A \in [-A_c - \Delta, -A_c + \Delta]$, applying the result in 3) we obtain

$$-\mathrm{Im}\,\lambda = \mathrm{Im}\,(-\lambda) \le \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c^T - A_c) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|)$$

and thereby also

$$\mathrm{Im}\,\lambda \ge \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \underline{\imath},$$

which concludes the proof. ∎

2 The bounds are exact in special cases

A real matrix A is called symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$. An interval matrix A^I is said to be symmetric if

$$A^{IT} = A^I$$

and skew-symmetric if

$$A^{IT} = -A^I,$$

where

$$A^{IT} = \{A^T;\ A \in A^I\}$$

and

$$-A^I = \{-A;\ A \in A^I\}.$$

Hence, $A^I = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if $[A_c^T - \Delta^T, A_c^T + \Delta^T] = [A_c - \Delta, A_c + \Delta]$, which is equivalent to symmetry of both A_c and Δ. Similarly, A^I is skew-symmetric if and only if A_c is skew-symmetric and Δ is symmetric.

Theorem 2 *The bounds (1.1) are exact (i.e., achieved over A^I) if A^I is symmetric and the bounds (1.2) are exact if A^I is skew-symmetric.*

Proof. 1) Let A^I be symmetric, so that A_c and Δ are symmetric. Since the continuous mapping $x \mapsto x^T A_c x + |x|^T \Delta |x|$ achieves its maximum over the unit sphere $\{x;\ \|x\|_2 = 1\}$, there exists an x satisfying

$$\overline{r} = x^T A_c x + |x|^T \Delta |x| \tag{2.1}$$

and $\|x\|_2 = 1$. Define a diagonal matrix S by

$$S_{jj} = \begin{cases} 1 & \text{if } x_j \geq 0, \\ -1 & \text{if } x_j < 0 \end{cases}$$

$(j = 1, \ldots, n)$, then $|x| = Sx$ and from (2.1) we have

$$\overline{r} = x^T A_c x + x^T S \Delta S x = x^T (A_c + S\Delta S) x \leq \lambda_{\max}(A_c + S\Delta S), \tag{2.2}$$

where $\lambda_{\max}(A_c + S\Delta S)$ denotes the maximal eigenvalue of $A_c + S\Delta S$ (which is symmetric since both A_c and Δ are symmetric). Since $|S\Delta S| = \Delta$, the matrix $A_c + S\Delta S$ belongs to A^I, hence

$$\lambda_{\max}(A_c + S\Delta S) \leq \overline{r}$$

by Theorem 1, which combined with (2.2) gives

$$\overline{r} = \lambda_{\max}(A_c + S\Delta S),$$

hence $\overline{r}$ is achieved over A^I (even more, it is achieved at a symmetric matrix in A^I, cf. Hertz [1]). The proof for $\underline{r}$ is analogous; in this case we obtain

$$\underline{r} = \lambda_{\min}(A_c - S\Delta S).$$

2) Let A^I be skew-symmetric, so that A_c is skew-symmetric and Δ is symmetric. We have

$$\bar{\iota} = x_1^T(A_c - A_c^T)x_2 + \Delta \circ |x_1x_2^T - x_2x_1^T| \tag{2.3}$$

for some x_1, x_2 satisfying $\|(x_1, x_2)\|_2 = 1$. Define

$$z_{ij} = \begin{cases} -1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j < 0, \\ 0 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j = 0, \\ 1 & \text{if } (x_1)_i(x_2)_j - (x_2)_i(x_1)_j > 0 \end{cases}$$

$(i, j = 1, \ldots, n)$, then $z_{ij} = -z_{ji}$ for each i, j, hence the matrix $\tilde{\Delta}$ defined by

$$\tilde{\Delta}_{ij} = z_{ij}\Delta_{ij}$$

$(i, j = 1, \ldots, n)$ is skew-symmetric (since Δ is symmetric). Let

$$A = A_c + \tilde{\Delta},$$

then $A \in A^I$ and A is skew-symmetric (since both A_c and $\tilde{\Delta}$ are skew-symmetric). Next, from (2.3) we have

$$\begin{aligned} \bar{\iota} &= x_1^T(A_c - A_c^T)x_2 + \sum_{ij} \Delta_{ij}z_{ij}(x_1x_2^T - x_2x_1^T)_{ij} \\ &= x_1^T(A_c - A_c^T)x_2 + \tilde{\Delta} \circ (x_1x_2^T - x_2x_1^T) \\ &= x_1^T(A_c - A_c^T)x_2 + x_1^T\tilde{\Delta}x_2 - x_2^T\tilde{\Delta}x_1 \\ &= x_1^TAx_2 - x_2^TAx_1 \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{aligned}$$

where the matrix

$$\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \tag{2.4}$$

is symmetric since A is skew-symmetric. Denote by λ the maximal eigenvalue of (2.4) (which is real), then from the above expression for $\bar{\iota}$ we have

$$\bar{\iota} \leq \lambda \tag{2.5}$$

and there exists a vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \neq 0$ satisfying

$$\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which implies

$$A(y_1 + y_2i) = -\lambda y_2 + \lambda i y_1 = \lambda i(y_1 + y_2i),$$

thus λi is an eigenvalue of A. Hence

$$\lambda \leq \bar{\iota}$$

by Theorem 1, which combined with (2.5) gives

$$\bar{\imath} = \lambda = \mathrm{Im}\,(\lambda i),$$

hence $\bar{\imath}$ is achieved as the imaginary part of an eigenvalue of a matrix in A^I. To prove an analogous result for $\underline{i}$, let us apply the result just proved to the interval matrix $[-A_c - \Delta, -A_c + \Delta]$, which is also skew-symmetric. Then we have

$$\mathrm{Im}\,\lambda = \max_{\|(x_1,x_2)\|_2=1} (x_1^T(A_c^T - A_c)x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|)$$

for an eigenvalue λ of some $\tilde{A} \in [-A_c - \Delta, A_c + \Delta]$, hence

$$\mathrm{Im}\,(-\lambda) = -\mathrm{Im}\,\lambda = \min_{\|(x_1,x_2)\|_2=1} (x_1^T(A_c - A_c^T)x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) = \underline{i}$$

for the eigenvalue $-\lambda$ of $-\tilde{A} \in [A_c - \Delta, A_c + \Delta]$, which shows that $\underline{i}$ is achieved as well. ■

3 Practical bounds

Theorem 3 *Let $A^I = [A_c - \Delta, A_c + \Delta]$ be a square interval matrix. Then for each eigenvalue λ of each $A \in A^I$ we have*

$$\lambda_{\min}(A_c') - \lambda_{\max}(\Delta') \leq \mathrm{Re}\,\lambda \leq \lambda_{\max}(A_c') + \lambda_{\max}(\Delta'), \tag{3.1}$$

$$\lambda_{\min}(A_c'') - \lambda_{\max}(\Delta'') \leq \mathrm{Im}\,\lambda \leq \lambda_{\max}(A_c'') + \lambda_{\max}(\Delta''), \tag{3.2}$$

where

$$\begin{aligned}
A_c' &= \frac{1}{2}(A_c + A_c^T), \\
\Delta' &= \frac{1}{2}(\Delta + \Delta^T), \\
A_c'' &= \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix}, \\
\Delta'' &= \begin{pmatrix} 0 & \Delta' \\ \Delta' & 0 \end{pmatrix}.
\end{aligned}$$

Comments. $\lambda_{\min}, \lambda_{\max}$ denote the minimal and maximal eigenvalue of a symmetric matrix, respectively. Notice that all the matrices $A_c', \Delta', A_c'', \Delta''$ are symmetric by definition. Since $\lambda_{\max}(D) = \varrho(D)$ (spectral radius) holds for a nonnegative symmetric matrix D, the formulae (3.1), (3.2) may also be written in the form

$$\lambda_{\min}(A_c') - \varrho(\Delta') \leq \mathrm{Re}\,\lambda \leq \lambda_{\max}(A_c') + \varrho(\Delta'),$$

$$\lambda_{\min}(A_c'') - \varrho(\Delta'') \leq \mathrm{Im}\,\lambda \leq \lambda_{\max}(A_c'') + \varrho(\Delta'').$$

Proof. Let λ be an eigenvalue of a matrix $A \in A^I$.

1) Since

$$\begin{aligned}
\overline{r} &= \max_{\|x\|_2=1} (x^T A_c x + |x|^T \Delta |x|) \\
&\leq \max_{\|x\|_2=1} x^T A_c x + \max_{\|x\|_2=1} |x|^T \Delta |x| \\
&= \max_{\|x\|_2=1} x^T A'_c x + \max_{\|x\|_2=1} |x|^T \Delta' |x| \\
&= \lambda_{\max}(A'_c) + \lambda_{\max}(\Delta'),
\end{aligned}$$

by Theorem 1 there holds

$$\operatorname{Re} \lambda \leq \lambda_{\max}(A'_c) + \lambda_{\max}(\Delta'),$$

which is the right-hand side inequality in (3.1).

2) The proof of the left-hand side inequality is analogous since

$$\underline{r} \geq \min_{\|x\|_2=1} x^T A_c x - \max_{\|x\|_2=1} |x|^T \Delta |x| = \lambda_{\min}(A'_c) - \lambda_{\max}(\Delta').$$

3) We have

$$\begin{aligned}
\overline{\imath} &= \max_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 + \Delta \circ |x_1 x_2^T - x_2 x_1^T|) \\
&\leq \max_{\|(x_1,x_2)\|_2=1} (x_1^T A_c x_2 - x_2^T A_c x_1) + \max_{\|(x_1,x_2)\|_2=1} (|x_1|^T \Delta |x_2| + |x_2|^T \Delta |x_1|) \\
&= \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & A_c \\ -A_c & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix}^T \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \\
&= \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\quad + \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(\Delta + \Delta^T) \\ \frac{1}{2}(\Delta + \Delta^T) & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \\
&= \lambda_{\max}(A''_c) + \lambda_{\max}(\Delta'').
\end{aligned}$$

Hence Theorem 1 gives

$$\operatorname{Im} \lambda \leq \overline{\imath} \leq \lambda_{\max}(A''_c) + \lambda_{\max}(\Delta''),$$

which is the right-hand side inequality in (3.2).

4) An analogous reasoning gives

$$\begin{aligned}
\underline{i} &= \min_{\|(x_1,x_2)\|_2=1} (x_1^T (A_c - A_c^T) x_2 - \Delta \circ |x_1 x_2^T - x_2 x_1^T|) \\
&\geq \min_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(A_c - A_c^T) \\ \frac{1}{2}(A_c^T - A_c) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\quad - \max_{\|(x_1,x_2)\|_2=1} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}(\Delta + \Delta^T) \\ \frac{1}{2}(\Delta + \Delta^T) & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} \\
&= \lambda_{\min}(A''_c) - \lambda_{\max}(\Delta''),
\end{aligned}$$

which in view of Theorem 1 implies the left-hand side inequality in (3.2). ∎

4 Consequences

We keep the notations $A'_c, \Delta', A''_c, \Delta''$ introduced in Theorem 3.

Corollary 1 *If*

$$\lambda_{\max}(A'_c) + \lambda_{\max}(\Delta') < 0,$$

then A^I is (Hurwitz) stable.

Proof. Indeed, in this case Theorem 3 implies $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of each $A \in A^I$. ∎

We note that a symmetric interval matrix may contain nonsymmetric matrices with complex eigenvalues. Similarly, a skew-symmetric interval matrix may contain matrices with nonzero real parts.

Corollary 2 *If A^I is symmetric, then*

$$|\operatorname{Im}\lambda| \leq \lambda_{\max}(\Delta'')$$

for each eigenvalue λ of each $A \in A^I$.

Proof. The result follows from (3.2) since $A''_c = 0$ in view of symmetry of A_c, hence $\lambda_{\min}(A''_c) = \lambda_{\max}(A''_c) = 0$. ∎

Corollary 3 *If A^I is skew-symmetric, then*

$$|\operatorname{Re}\lambda| \leq \lambda_{\max}(\Delta')$$

for each eigenvalue λ of each $A \in A^I$.

Proof. The assertion is a consequence of (3.1) since $A'_c = 0$. ∎

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