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Milan Daniel

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Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic
phone: (+422) 66414244 fax: (+422) 8585789
e-mail: milan@uivt.cas.cz

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Algebraic Structures Related to Dempster-Shafer Theory

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Abstract

There are described some algebraic structures on a space of belief functions on a two-element frame, namely so called Dempster's semigroup (with Dempster's operation $\oplus$), dempsteroids, and their basic properties. The present paper is devoted to the investigation of automorphisms of Dempster's semigroup. Full characterization of general and ordered automorphisms is obtained, their parametric description is stated both in intuitive and explicit forms. There is also full characterization of ordered endomorphisms, and other related results.

Keywords

Uncertainty, Belief Functions, Dempster-Shafer theory, Dempster's semigroup, Dempsteroid

[1] Major part of this text to appear in ADVANCES IN INTELLIGENT COMPUTING, edited by B.Bouchon-Meunier, R.R.Yager and L.A.Zadeh, Springer Verlag, 1995.

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1 Introduction

Dempster-Shafer (D-S) theory of belief functions (see [14]) is well recognized as one of the main directions in uncertainty management in AI. The first attempts at applying D-S theory in AI were related to MYCIN-like systems (see e.g. [6]), but now D-S theory is applied to uncertainty in general (see e.g. [15]).

Belief functions are numeric (quantitative) characteristics of belief. Investigation of their general properties is of great importance; the study of comparative properties (non-numeric comparison of belief) may serve as an example (cf. e.g. [3]). A related example is the study of algebraic properties of systems of belief functions, showing in particular the behaviour of important properties of belief functions under isomorphisms of the systems in question. Results of this kind are relevant to the attempts to make the user of a concrete AI-system (dealing with uncertainty) free from relying heavily on concrete numbers (weights, numerical beliefs, etc.), and free to make his/her decision dependent on more general and more stable things (e.g., things not dependent on transition from one belief structure to an isomorphic one - rather loosely worded).

Algebraic aspects were investigated by Hájek and Valdés [11] (for basic belief assignments on two-element frames), see also [9] chapter IX. We can also mention later papers axiomatizing, in an algebraic manner, Dempster's rule on general (finite) frames: [7, 12, 16].

The present paper is devoted to the investigation of automorphisms of the semigroup of (non-extremal) belief functions on a two-element frame (with Dempster's operation $\oplus$ - we refer to Dempster's semigroup). Full characterization is obtained and there are various related results. In particular, the question of existence of non-trivial automorphisms of Dempster's semigroup is answered positively: there are lots of them.

This paper has the following structure:

In the second section are briefly introduced used basic algebraic notions. The third section recalls main definitions and results from [11], in particular, the definition of Dempster's semigroup and of the class of algebras called dempsteroids.

Sections 4 and 5 deal with the standard dempsteroid $\mathbf{D}_0$ (Dempster's semigroup with an additional structure, the usual algebraic structure underlying belief functions on a two-element frame of discernment); the main result is full characteratization of all automorphisms of $\mathbf{D}_0$ (sect. 4) and also full characteratization of all of its o-endomorphisms (sect. 5). The latter result gives us a characteratization of all dempsteroids constructed by o-isomorphic embedding of $\mathbf{D}_0$ to itself.

The sixth section summarizes some conclusions for comparative (non-numeric) theories of uncertainty. Some tasks for future research in the theory of Dempster's semigroup are also mentioned.

2 Preliminaries

Let us recall some basic algebraic notions before we begin algebraic description of D-S theory.

A *commutative semigroup* (called also an *Abelian semigroup*) is a structure $X =$

$(X, \oplus)$ formed by set X and a binary operation $\oplus$ on X which is commutative and associative ($x \oplus y = y \oplus x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ holds for all $x, y, z \in X$). A *commutative group* is a structure $Y = (Y, \oplus, -, o)$ such that $(Y, \oplus)$ is a commutative semigroup, o is a neutral element ($x \oplus o = x$) and $-$ is a unary operation of inverse ($x \oplus -x = o$). An *ordered commutative semigroup* consists of a commutative semigroup Y as above and a linear ordering $\leq$ of its elements satisfying monotonicity ($x \leq y$ implies $x \oplus z \leq y \oplus z$ for all $x, y, z \in X$). A subset of Y which is a (semi)group itself is called a *sub(semi)group*.

A *homomorphism* $p : (X, \oplus_1) \longrightarrow (Y, \oplus_2)$ is a mapping which preserves structure, i.e. $p(x \oplus_1 y) = p(x) \oplus_2 p(y)$ for each $x, y \in X$. The special cases are *automorphisms*, which are bijective morphisms from a structure onto itself; *endomorphisms* are morphisms to a substructure of the original one. Morphisms which also preserve ordering of elements are called *ordered morphisms* (*o-automorphisms, o-endomorphisms*, ...).

Ordered structures and ordered morphisms are very important for a comparative approach to uncertainty management.

3 On Dempster's semigroup

Now we introduce some principal notions according to [9]. For a two-valued frame $\Theta = \{0, 1\}$ each basic belief assignment determines a d-pair $(m(1), m(0))$ and conversely, each d-pair determines a basic belief assignment:

Definition 3.1 *A Dempster pair (or d-pair) is a pair of reals such that $a, b \geq 0$ and $a + b \leq 1$. A d-pair (a, b) is Bayesian if $a + b = 1$, (a, b) is simple if $a = 0$ or $b = 0$, in particular, extremal d-pairs are pairs (1,0) and (0,1). (Definitions of Bayesian and simple d-pairs correspond evidently to the usual definitions of Bayesian and simple belief assignments [9, 14]).*

Definition 3.2 *Dempster's semigroup $D_0 = (D_0, \oplus)$ is the set of all non extremal Dempster pairs, endowed with the operation $\oplus$ and two distinguished elements $0 = (0, 0)$ and $0' = (\frac{1}{2}, \frac{1}{2})$, where the operation $\oplus$ is defined by*

$$(a, b) \oplus (c, d) = (1 - \frac{(1-a)(1-c)}{1-(ad+bc)},\ 1 - \frac{(1-b)(1-d)}{1-(ad+bc)}).$$

Remark It is well known that this operation corresponds to Dempster's rule of combination of basic belief assignments.

Definition 3.3 *For $(a, b) \in D_0$ let*

$$\begin{aligned}
-(a, b) &= (b, a), \\
h(a, b) &= (a, b) + 0' = (\tfrac{1-b}{2-a-b},\ \tfrac{1-a}{2-a-b}), \\
h_1(a, b) &= \tfrac{1-b}{2-a-b}, \\
f(a, b) &= (a, b) \oplus (b, a) = (\tfrac{a+b-a^2-b^2-ab}{1-a^2-b^2},\ \tfrac{a+b-a^2-b^2-ab}{1-a^2-b^2}).
\end{aligned}$$

For $(a,b),(c,d) \in D_0$ we define
$(a,b) \leq (c,d)$ *iff* $h_1(a,b) < h_1(c,d)$ *or if* $h_1(a,b) = h_1(c,d)$ *and* $a \leq c$.

Let G denote the set of all Bayesian nonextremal d-pairs. Let us denote the set of all simple d-pairs such that $b = 0$ $(a = 0)$ as S_1 (S_2). Furthermore, put $S = \{(a,a) :\leq a \leq 0.5\}$.
(Note: $h(a,b)$ is an abbreviation for $h((a,b))$, etc.)

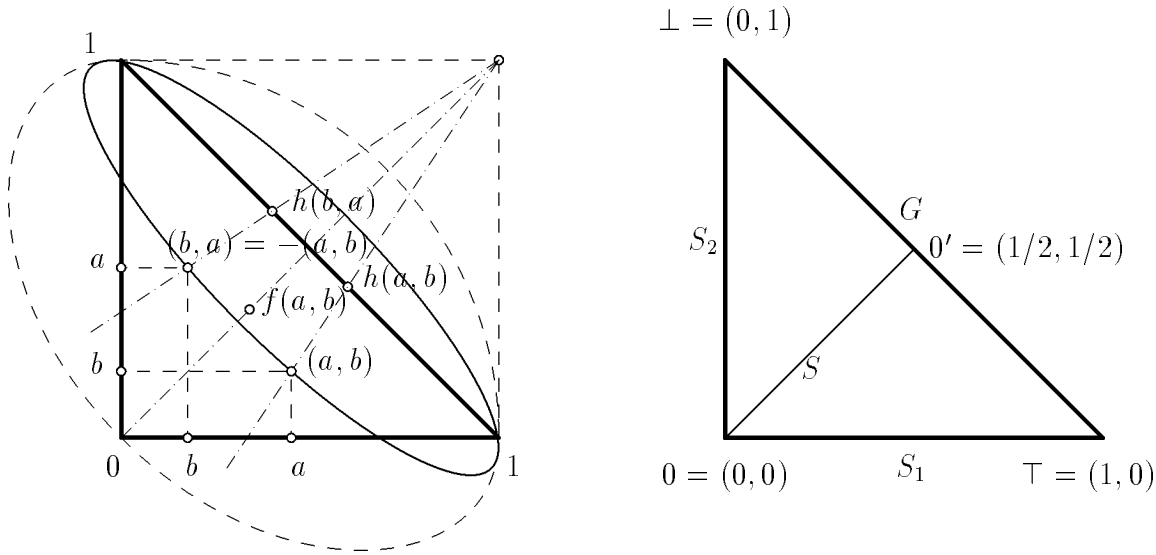

Figure 3.1: **Dempster's semigroup.** Homomorphism h is, in this representation, a projection to group G along the straight lines running through the point $(1,1)$. All Dempster pairs laying on the same ellipse are, by homomorphism f, mapped to the same d-pair in semigroup S.

Theorem 3.4

(i) Dempster's semigroup with the relation $\leq$ is an ordered commutative semigroup with neutral element 0; $0'$ is the only nonzero idempotent of it.

(ii) The set G with ordering $\leq$ is an ordered Abelian group $(G, \oplus, -, 0', \leq)$ which is isomorphic to the PROSPECTOR group **PP** *(cf. [9]) and consequently isomorphic to the additive group of reals with usual ordering.*

(iii) The sets S, S_1 and S_2 with the operation $\oplus$ and the ordering $\leq$ form ordered commutative semigroups with neutral element 0, and are all isomorphic to the semigroup of nonnegative elements of the MYCIN group **MC**.

(iv) The mapping h is an ordered homomorphism of the ordered Dempster's semigroup onto its subgroup G (i.e. onto **PP***).*

(v) The mapping f is a homomorphism of Dempster's semigroup onto its subsemigroup S (but it is not an ordered homomorphism).

For proofs see [9, 11, 17].

Remark PROSPECTOR group **PP** or $\mathbf{PP}_0$ is an OAG **PP**$= ((-1,1), \oplus_{PP}, -, \leq)$ or an OAG $\mathbf{PP}_0 = ((0,1), \oplus_{PP_0}, 1-x, \leq)$, where operations $\oplus_{PP}$, $\oplus_{PP_0}$ are defined by $x \oplus_{PP} y = \frac{x+y}{1+xy}$ and $x \oplus_{PP_0} y = \frac{xy}{xy+(1-x)(1-y)}$, $-$ and $\leq$ are usual operation and ordering of real numbers. $\oplus_{PP_0}$ is just the way of combination in a classical expert system PROSPECTOR.

Similarly MYCIN group **MC** is an OAG **MC**$= ((-1,1), \oplus_{MC}, -, \leq)$, where $x \oplus_{MC} y = x+y-xy$ for $x, y \geq 0$, $x \oplus_{MC} y = x+y = xy$ for $x, y \leq 0$, and $x \oplus_{MC} y = \frac{x+y}{1-min(|x|,|y|)}$ for $xy \leq 0$, $-$ and $\leq$ as usually. $\oplus_{MC}$ is just the way of combination in a classical expert system shell EMYCIN.

All the groups **PP**, $\mathbf{PP}_0$ and **MC** are mutually isomorphic.

Definition 3.5 *A dempsteroid is an algebra* $\mathbf{D} = (D, \oplus, \ominus, o, o', \leq)$ *satisfying the following:*

1. $(D, \oplus, o, \leq)$ *is an ordered commutative semigroup with* o *as a neutral element,*

2. $\ominus(\ominus x) = x$,

   $\ominus(x \oplus y) = (\ominus x) \ \oplus \ (\ominus y) \qquad$ for each $\quad x, y \in D$,

3. $o \leq o'$,

4. *for each* $x \in D$: $\ o \leq x \leq o'$ *iff* $x \oplus o' = o'$ *iff* $x = \ominus x$, *the set of all* x *satisfying any of these conditions is denoted by* S.

5. *for each* $x, y \in S$ *such that* $x \leq y$ *there is* $z \in S$ *such that* $x \oplus y = z$ *(subtraction in* S*).*

Let us define mappings h *and* f *on dempsteroid* D: $h(x) = x \oplus o'$, $f(x) = x \oplus (\ominus x)$ *and let us denote* G *the set* $G = \{x \oplus o' : x \in D\}$.

Definition 3.6 *The standard dempsteroid* $\mathbf{D}_0 = (D_0, \oplus, -, 0, 0', \leq)$ *is the dempsteroid defined by Dempster's semigroup, by the operation* $-$, *and by the ordering* $\leq$, *see definition 3.*

In order to use dempsteroids in uncertain information processing, it is necessary to enrich the defined algebraic structures with extremal elements:

Definition 3.7 *An extended dempsteroid* $\mathbf{D}^+ = (D \cup \{\bot, \top\}, \oplus, \ominus, o, o', \leq)$ *is an algebraic structure resulting from taking a dempsteroid and adding extremal elements* $\bot, \top$ *in the following way:*

$$\begin{array}{lll} x \oplus \bot = \bot & \text{and} \quad x \oplus \top = \top & \text{for all } x \in D, \\ \ominus\bot = \top & \text{and} \quad \ominus\top = \bot, & \\ \bot \leq x \leq \top & & \text{for all } x \in D. \end{array}$$

For the standard dempsteroid let us define $\bot = (0,1)$, $\top = (1,0)$.

4 Automorphisms of Dempster's semigroup

It is useful to summarize features of potential nontrivial automorphisms of Dempster's semigroup before attempting to find them.

In [1] it is proved for any automorphism g of Dempster's semigroup (as a semigroup in the simple pure algebraic sense $(D_0, \oplus)$) that g preserves d-pairs 0 and $0'$, and that g commutes with the homomorphism h. Further, it is proved that g commutes with the operation $-$ (see 3.3) if and only if it commutes with the homomorphism f.

In order to stress that g commutes with $-$, we shall say that g is an automorphism of the standard dempsteroid:

Definition 4.1 *An automorphism of the standard dempsteroid is a one-to-one mapping* $\mathbf{D}_0$ *onto* $\mathbf{D}_0$ *which preserves the elements* 0 *and* $0'$ *and commutes with the operations* $\oplus, -$. *If it preserves the ordering* $\leq$ *of* $\mathbf{D}_0$, *we call it an o-automorphism of* $\mathbf{D}_0$.

When looking for nontrivial automorphisms of $\mathbf{D}_0$ it is very important to show preserving of subalgebras G and S of $\mathbf{D}_0$ by all its automorphisms, and preserving of subgroups S_1 and S_2 of $\mathbf{D}_0$ by an arbitrary o-automorphism of the standard dempsteroid.

For description of o-automorphisms of the standard dempsteroid it is useful to describe all the o-automorphisms of its subgroup G and of its subsemigroups S, S_1, and S_2, too.

Proposition 4.2 *(i) Any automorphism* g *of* $\mathbf{D}_0$ *maps elements of subgroup* G *(subsemigroup* S*) onto the subgroup* G *(subsemigroup* S*).*

(ii) Any o-automorphism g *of* $\mathbf{D}_0$ *maps elements of subsemigroups* S_1 *and* S_2 *onto the subsemigroups* S_1 *and* S_2, *respectively.*

Idea of a proof: For any automorphism of Dempster's semigroup it is easy to verify that it maps elements of the subalgebras G (or S) back to G (or S). For o-automorphisms it is possible to show the preservation of the subsemigroups S_1 and S_2 too. (For contradiction we can presume a value $g(a, 0) = (c, d)$, $d \neq 0$.)

For a particular proof of this proposition and for particular proofs of other following statements see [1].

Proposition 4.3 *O-automorphisms of the ordered Abelian group* G *and of the ordered Abelian semigroups* S, S_1 *and* S_2 *are the just following mappings:*

$$\begin{array}{rcrcll} G & : & g_G(x) & = & \frac{x^k}{x^k+(1-x)^k}, & k > 0 \qquad (\mathbf{G}) \\ S & : & g_S(x) & = & \frac{(1-2x)^l-(1-x)^l}{(1-2x)^l-2(1-x)^l}, & l > 0 \qquad (\mathbf{S}) \\ S_1, S_2 & : & g_M(x) & = & 1-(1-x)^m, & m > 0 \qquad (\mathbf{M}). \end{array}$$

Proof: Idea of the proof is based on isomorphisms of the algebras G, S, S_1 and S_2 onto the PROSPECTOR or MYCIN groups **PP** and **MC** and succeeding isomorphisms onto $\mathbf{Re} = (Re, +, 0, \leq)$, see e.g. [9]. We know all the o-automorphisms of $\mathbf{Re}$ ($x \to cx$, $c > 0$, see [5]) and so we can isomorfically transform them onto G, S, S_1 and S_2. Corresponding o-isomorphic transformations are:

$$\begin{array}{lcll} \mathbf{Re} & \to & G: & a = \dfrac{z^x}{1+z^x}, \\ \mathbf{Re}_{\geq 0} & \to & S - \{0'\}: & s = \dfrac{z^x - 1}{2 \cdot z^x - 1}, \quad x \geq 0, \\ \mathbf{Re}_{\geq 0} & \to & S_1, S_2: & a = \dfrac{z^x - 1}{z^x}, \quad x \geq 0. \end{array}$$

From proposition 4.2.(i) follows that the restriction of any o-automorphism of the standard dempsteroid to its subgroup G (subsemigroup S, S_1, S_2) is an o-automorphism of the subgroup G (subsemigroup S, S_1, S_2); i.e. from preservation of the subalgebras G, S, S_1 and S_2 of $\mathbf{D}_0$ it is seen that any o-automorphism g of $\mathbf{D}_0$ induces o-automorphisms of these subalgebras of $\mathbf{D}_0$.

On the other hand we can try to use o-automorphisms of these subalgebras to express o-automorphism g of the standard dempsteroid. For this reason we shall use the following lemmata.

Lemma 4.4 *Any o-automorphism g of $\mathbf{D}_0$ has the following property:*

$$\begin{aligned} g(a,b) \in f^{-1}(g(f(a,b))) \cap h^{-1}(g(h(a,b))) = & \qquad (\mathbf{g}\cap) \\ = \{(x,y) \quad : \quad & f(x,y) = (g_S(p_i(f(a,b))), g_S(p_i(f(a,b)))) \\ \& \quad & h(x,y) = (g_G(p_1(h(a,b))), g_G(p_2(h(a,b))))\}, \end{aligned}$$

where g_G and g_S are o-automorphisms of the subalgebras G and S induced by g and for $i = 1, 2$ p_i is projection to the i-th coordinate.

The proof of the lemma follows from commutativity of o-automorphisms of the standard dempsteroid with the homomorphisms h and f. h^{-1} denotes a set of preimages of a point (a, b) with respect to the homomorphism h, similarly for f and $f^{-1}(a, b)$. For particularities see [1].

According to lemma 11 we can express any o-automorphism of $\mathbf{D}_0$ by a pair of o-automorphisms g_G and g_M of subalgebras G and S. Now, we can ask the principial question: "Which pairs of o-automorphisms g_G and g_S define an o-automorphism of $\mathbf{D}_0$?". It would be easy in the case of looking for the o-automorphisms of the structure $G \times S$, where every such a pair defines by ($\mathbf{g}\cap$) an o-automorphisms of the global structure. The situation is more complicated in the case of $\mathbf{D}_0$.
E.g.: Let $g_S = 1_S$ and g_G in the form **(G)** (i.e., g_S is in the form **(S)**, where $l = 1$). If $k > 1$ then g defined by ($\mathbf{g}\cap$) maps elements of S_i out of $\mathbf{D}_0$. In the case of $k < 1$, g defined by ($\mathbf{g}\cap$) maps elements of S_i into a proper subset of $\mathbf{D}_0$. So neither the first nor the second example of pair of o-automorphisms does not define by ($\mathbf{g}\cap$) any o-automorphism of $\mathbf{D}_0$, because they both violate 4.2.(ii).

Here is seen the principial importance of a preserving of the subsemigoup S (see proposition 4.2.(i)), although no o-automorphism g_M is explicitly included in the expression (**g**$\cap$).

We have the biparametrical description of potential o-automorphisms of $\mathbf{D}_0$ with parameters k and l, $k, l > 0$, see **(G)**, **(S)**. Now, there is a question to solve: "Is there any pair $(k, l) \neq (1, 1)$ which defines an o-automorphism of $\mathbf{D}_0$?", and if Yes: "What is the set of all such a pairs ?" The answer follows from the next lemmata.

Lemma 4.5 *(Necessary conditions for g_G and g_S)*

($\circ$) A pair of o-automorphisms g_G and g_M induced by the same o-automorphism g of the standard dempsteroid $\mathbf{D}_0$ satisfies: $k = m$.

($\circ$) A pair of o-automorphisms g_S and g_M induced by the same o-automorphism g of the standard dempsteroid $\mathbf{D}_0$ satisfies: $l = m$.

Here k, l, m are constants from expressions **(G)**, **(S)**, **(M)**.

Conclusion:

($\bullet$) A pair of o-automorphisms g_G and g_S induced by the same o-automorphism g of the standard dempsteroid $\mathbf{D}_0$ satisfies: $k = l$.

Lemma 4.5 says the very important fact that all the o-automorphisms g_G, g_S and g_M induced by the same o-automorphism of $\mathbf{D}_0$ are defined by the same constant from the expressions **(G)**, **(S)**, **(M)**.

Idea of a proof: To prove the lemma, one heavily uses the preservation of subsemigroups S_1 and S_2 by any o-automorphism of $\mathbf{D}_0$, see proposition 4.2.(ii).

To prove $k = m$ we use commutativity of an o-automorphism g with the homomorphism h and preservation of the subsemigroup S_1 by g. It must be $h(g(a, 0)) = g(h(a, 0))$. Let us skip the computation of both sides of the equation. Similarly from commutativity of g and f and preservation of S_2 by g it is possible show that $l = m$, thus $k = l$.

Before saying of the following lemma, let us suppose an extension of homomorphisms f and h to the Cartesian Product $G \times S \supset \mathbf{D}_0$ (it allows easier formulation of it). In Fig. 1. $G \times S$ is an area bordered by group G and by greater, dashed ellipse.)

Lemma 4.6 *(Properties of mapping defined by the condition* (**g**$\cap$)*)*
Let g_G is an o-automorphism of the subgroup G and g_S is an o-automorphism of the subsemigroup S of $\mathbf{D}_0$.
It holds:

(i) Expression (**g**$\cap$) *defines one-to-one mapping of $G \times S$ onto $G \times S$.*

(ii) An inverse mapping g^{-1} of g defined by $(\mathbf{g}\cap)$ *with o-automorphisms g_G and g_S (of the subalgebras G and S) is also a mapping defined according to* $(\mathbf{g}\cap)$*, namely with the o-automorphisms g_G^{-1} and g_S^{-1}.*

(iii) A mapping defined according to $(\mathbf{g}\cap)$ *maps elements of the subalgebras G and S in order back onto G and S.*

(iv) For elements of the subalgebras G and S and a mapping g defined by $(\mathbf{g}\cap)$ *holds*

$$\begin{aligned} g(a, 1-a) &= (g_G(a,a), g_G(1-a, 1-a)), \\ g(a,a) &= (g_S(a,a), g_S(a,a)). \end{aligned}$$

(v) A mapping g defined by $(\mathbf{g}\cap)$ *commutes with the homomorphisms h and f of $\mathbf{D}_0$.*

Lemma 4.7 *(Sufficient condition for g_G and g_S) If the mapping defined according to* $(\mathbf{g}\cap)$ *satisfies $k = l$ then it maps elements of algebras S_1 and S_2 onto S_1 and S_2 respectively.*

Now we can formulate the theorem on o-automorphisms of the standard dempsteroid.

Theorem 4.8 (On o-automorphisms) *O-automorphisms of the standard dempsteroid are just the mappings defined according to* $(\mathbf{g}\cap)$ *by pairs of o-automorphisms of the subgroup G and of the subsemigroup S such that*

$$\begin{aligned} g_G(x) &= \frac{x^k}{x^k + (1-x)^k}, \\ g_S(x) &= \frac{(1-2x)^l - (1-x)^l}{(1-2x)^l - 2(1-x)^l}, \end{aligned}$$

where $k = l > 0$.

Idea of a proof: We have seen (lemma 4.5), that equality $k = l$ is the necessary condition for o-automorphisms g_G and g_S induced by an o-automorphism g of the standard dempsteroid $\mathbf{D}_0$ and that all o-automorphisms of $\mathbf{D}_0$ induce such a g_G and g_S.

For a finishing of the proof we have to verify that every mapping g, defined according to the theorem, preserves the operation $\oplus$.

Lemma 4.6 says that any such a mapping g from the theorem has other properties of o-automorphisms of $\mathbf{D}_0$ inclusive preserving of subsemigroups S_1 and S_2, see lemma 4.7, i.e. it is an o-automorphism of $\mathbf{D}_0$.

Corollary 4.9 *There is just one extension of any o-automorphism of the subgroup G to an o-automorphism of the whole $\mathbf{D}_0$.*

The theorem on o-automorphisms both says that there exist nonidentical o-automorphisms of the standard dempsteroid and describes the method for determining and of constructing of all of them. The explicit form of o-automorphisms of $\mathbf{D}_0$ is expressed in the following theorem, according to which it is possible to compute all the functional values for any o-automorphism g of $\mathbf{D}_0$.

Theorem 4.10 (On explicit o-automorphisms) *O-automorphisms of the standard dempsteroid are just mappings in the form*

$$g(a,b) \;=\; (A,B),$$

where

$$\begin{aligned} A &= \tfrac{P_0^{2c}-(1-b)^cQ+(1-a)^cP_0^c}{P_0^{2c}-Q^2}, \\ B &= 1-(1-A)\cdot\tfrac{(1-b)^c}{(1-a)^c} \;=\; 1-\tfrac{(1-b)^c}{(1-a)^c}\cdot\tfrac{-Q^2+(1-b)^cQ-(1-a)^cP_0^c}{P_0^{2c}-Q^2}, \\ P_0 &= 1-(a+b), \\ Q &= (1-a)^c+(1-b)^c, \qquad\qquad c>0. \end{aligned}$$

Idea of a proof: Let $g(a,b) = (A,B)$. We can express a value $h(A,B)$ from the definition of the homomorphism h and from the expression ($\mathbf{g}\cap$). From the resulting equation we can express B. Analogously from both the expression of $f(A,B)$ and the expression for B we can express A. The rest is tedious computing.

It is possible to make a non-trivial generalization of the theorem 4.8 to general (nonordered) automorphisms of $\mathbf{D}_0$ as 4.11. From the theorems 4.10 and 4.11 we can derive the explicit form of general automorphisms of $\mathbf{D}_0$.

Theorem 4.11 (On automorphisms) *Automorphisms of the standard dempsteroid are just mappings $g(a,b) = g_0(a,b)$ or $g(a,b) = -g_0(a,b)$, where $g_0(a,b)$ is an o-automorphism of $\mathbf{D}_0$.*

A proof of the theorem is similar as for o-automorphisms, but it is a little bit more complicated than in the case of o-automorphisms because an automorphism of $\mathbf{D}_0$ generally does not preserve the semigroups S_i. So it is necessary to prove that a general automorphism of $\mathbf{D}_0$ maps S_1 (S_2) either onto itself or onto S_2 (S_1).

In this case we get for constants k and l determining g_G and g_S the following condition $k = l \vee k = -l$, $l > 0$.

Theorem 4.12 (On fixed points of o-automorphisms) *Every non-identical o-automorphism of the extended standard dempsteroid has just the four fixed points, namely (0,0), (0,1), ($\frac{1}{2}$,$\frac{1}{2}$) and (1,0).*

Proof: It is easy to verify that the stated points are fixed points of o-automorphisms of $\mathbf{D}_0^+$. For contradiction we further presume that there is an another fixed point (a,b) of an o-automorphism g of $\mathbf{D}_0^+$. Thus $h(a,b)$ is a fixed point of the corresponding g_G, so it must be $h(a,b) = (\frac{1}{2},\frac{1}{2})$ and $(0,0) \neq (a,b) \neq \frac{1}{2},\frac{1}{2})$ should be a fixed point of the corresponding g_S, it is a contradiction.

Corollary 4.13 *A general automorphism of* $\mathbf{D}_0^+$ *has either four fixed points (as o-automorphisms) or two fixed points (0,0) and* $(\frac{1}{2},\frac{1}{2})$ *(in the case of the violation of the ordering).*

5 Endomorphisms of Dempster's semigroup

Let us investigate the case of endomorphisms now. We can use similar construction as in the case of automorphisms, thus we generalize the expression ($\mathbf{g}\cap$) to ($\mathbf{g}\cap'$) by admitting endomorphisms g_G and g_S of the subalgebras G and S.

So we get either a pair of automorphisms g_G and g_S, or a pair of two nonbijective endomorphisms, or a pair formed by one automorphism (g_G or g_S) and one nonbijective endomorphism (the remaining one). Similarly, as in the case of automorphisms, such a pair does not generally define an endomorphism of $\mathbf{D}_0$. For example, endomorphisms g_1 or g_2 of the subgroup S (where $g_1(x) \equiv 0$, $g_2(0') = 0'$, $g_2(x) = 0$ otherwise) with any automorphism g_G of the subgroup G does not define it.

The situation is fully described by the following two theorems.

Theorem 5.1 (On endomorphisms) *An endomorphism of the standard dempsteroid is defined according to* ($\mathbf{g}\cap'$) *by one of the following possibilities:*

1. *by a pair of automorphisms* g_G *and* g_S *of the subgroup* G *and the subsemigroup* S

   $g(x) = \frac{x^k}{x^k+(1-x)^k}$, $\quad g(x) = \frac{(1-2x)^l-(1-x)^l}{(1-2x)^l-2(1-x)^l}$, *where* $l \geq k \geq -l$, $l > 0$, $k \neq 0$.

2. *by an automorphism* g_G *of the subgroup* G *and by a non-bijective endomorphism* g_S *of the subsemigroup* S *satisfying:*

   $$\begin{aligned} & g_S(x) = 0' \quad \forall x \in S \\ \textit{or} \quad & g_S(0) = 0 \text{ and } g_S(x) = 0' \text{ for } \quad x \in S - \{0\}. \end{aligned}$$

3. *by an automorphism* g_S *of subgroup* S *and by the endomorphism* $G \to \{0'\}$ *of the subgroup* G.

4. *by a pair of non-bijective endomorphisms of the subalgebras* G *and* S,

   i.e. we have the following endomorphisms of the Dempster's semigroup:

   $$\begin{aligned} g : \mathbf{D}_0 &\to \{0\}, \\ g : \mathbf{D}_0 &\to \{0'\}, \\ g : \mathbf{D}_0 &\to \{0, 0'\}, \text{ by the both way : i.e.} \end{aligned}$$

   $$\begin{aligned} & g_1(0) = 0, \quad g_1(x) = 0' \quad \textit{for} \quad x \in S - \{0\} \qquad \textit{or} \\ & g_2(0') = 0', \quad g_2(x) = 0 \quad \textit{for} \quad x \in S - \{0'\} \ . \end{aligned}$$

Theorem 5.2 (On *o*-endomorphisms) *An o-endomorphism of the standard dempsteroid is defined according to* ($\mathbf{g} \cap '$) *by one of the following possibilities:*

1. *by a pair of o-automorphisms g_G and g_S of the subgroup G and the subsemigroup S*

   $g(x) = \frac{x^k}{x^k+(1-x)^k}, \quad g(x) = \frac{(1-2x)^l-(1-x)^l}{(1-2x)^l-2(1-x)^l}, \quad$ *where* $l \geq k > 0$.

2. *by an o-automorphism g_G of the subgroup G and by an non bijective o-endomorphism g_S of the subsemigroup S satisfying:*

   $$\begin{array}{ll} & g_S(x) = 0' \quad \forall x \in S \\ or & g_S(0) = 0 \quad \text{and} \quad g_S(x) = 0' \quad \text{for } x \in S - \{0\}. \end{array}$$

3. *following trivial o-endomorphisms of Dempster's semigroup (defined by a pair of non-bijective endomorphisms):*

   $$\begin{array}{rcl} g : \mathbf{D}_0 & \rightarrow & \{0\}, \\ g : \mathbf{D}_0 & \rightarrow & \{0'\}. \end{array}$$

At this moment, we are close to a theorem on subdempsteroids; it follows from the following:

Theorem 5.3 *Every o-endomorphism of the standard dempsteroid $\mathbf{D}_0$ defined by* $(\mathbf{g}\cap)$, *i.e., by a pair of o-automorphisms, where $l > k > 0$, is a bijective mapping from $\mathbf{D}_0$ onto a proper subdempsteroid of $\mathbf{D}_0$.*

Corollary 5.4 (Theorem on subdempsteroids) *Every pair of numbers k and l, where $l > k > 0$, uniquely defines a proper subdempsteroid $D_{[k,l]}$ of the standard dempsteroid $\mathbf{D}_0 = D_{[1,1]}$ and a bijective o-isomorphism between $D_{[1,1]}$ and $D_{[k,l]}$.*

6 Concluding remarks

6.1 Ordered morphisms of Dempster's semigroup and processing of uncertainty

A motivation of an algebraization of D-S theory came from rule-based systems managing uncertain rules. So the stated results and o-automorphisms of Dempster's semigroup are useful in rule-based knowledge systems [1], but their utilization is not at all restricted to field of rule-based systems.

We can utilize o-automorphisms of Dempster's semigroup everywhere we have discrete user scale of degrees of uncertainty and uncertainty management based on D-S theory. In that case, it is necessary to realize discrete degrees and intervals of them into continuous structure, into Dempster's semigroup. O-automorphisms of Dempster's semigroup give classes of equivalence of such realizations.

Discrete user scales of degrees of uncertainty is used, e.g., in expert systems AL/X [13] and EQUANT [8]. A discrete scale could be displayed to a user e.g. in the following form: False, almost false, a little false, maybe, a little true, almost true, true.

A realization of discrete degrees and their intervals into Dempster's semigroup is a special case of their realization into a general dempsteroid. Theoretical results on

subdempsteroids and on endomorphisms of Dempster's semigroup are useful for such a realization.

The theorem on fixpoints (4.12) is a helpful tool for a theoretical studying of such realizations too.

6.2 Perspectives for future investigation

The following are *open problems* of the theory of morphisms of Dempster's semigroup:

The examining of the existence of an automorphism of the $(D_0, \oplus)$ which violates the operation $-$ and finding a form of a such an automorphism. Or proving of the theorem that every automorphism of $(D_0, \oplus)$ preserves the operation $-$, so that it is also an automorphism of the standard dempsteroid $(D_0, \oplus, -, 0, 0', \leq)$. It does mean to discover if it is necessary to distinguish between automorphisms of Dempster's semigroup and automorphisms of the standard dempsteroid.

Further it is possible to investigate which other subalgebras are preserved by o-automorphisms of the standard dempsteroid besides G, S and S_i.

Another interesting problem is the question whether $D_{[k,l]} = D_{[m,n]}$ for $\frac{k}{l} = \frac{m}{n}$, and what is the relation of isomorphisms from these dempsteroids to $\mathbf{D}_0$, etc.

The great challenge for future research is the generalization of the algebraic structure for D-S with a general finite frame of discernment. The general algebraic structure will be much more complicated. A complexity of the structure is dependent on the number of various focal elements $\sim 2^n - 1$, where n is a number of elements of a used frame.

After description of the structure it will be necessary to try to define reasonable ordering on it, and study its o-automorphisms. This will be a very interesting research task.

6.3 Conclusions

An algebraic analysis of morphisms of Dempster's semigroup were presented. It contributes to recognition of the fact that algebraic structures related to Dempster-Shafer (D-S) theory are mathematically very rich and interesting. A complete parametric description of (o-)automorphisms (and o-endomorphisms as well) of the standard dempsteroid has been given.

The results on ordered morphisms are related to the investigation of comparative properties of belief functions (i.e., comparison of beliefs, not assigning exact numerical values), since comparative properties are preserved by o-automorphisms (and some of them even by o-endomorphisms). This should be related to the work by Dubois [3].

D-S theory with a two-element frame of discernment is useful in the case of processing of uncertainty expressed by discrete finite scale, so called degrees of belief, degrees of plausibility. And described o-automorphisms are very principal within the proces of realization of these degrees of uncertainty (resp. intervals of them) in the continuous Dempster's semigroup. The structure on the space of these intervals is in some sense related to representation of uncertainty by bilattices in the works by Esteva-Garcia-Godo [4].

The presented results can be also used as a very important basis for a description and a study of an algebraical structure of D-S theory with a general finite frame. As it has been mentioned, such a generalization would be much more complicated, and the structure of Dempster's semigroup would be many times embeded in a resulting generalized algebraic structrure.

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