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2013

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Technical report No. V-1185

09.07.2013

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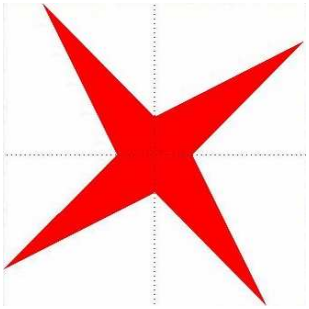
Abstract:

To two existing sufficient conditions for regularity of interval matrices we add a third one and then we merge all three into a single triple sufficient condition.[1]



Keywords:
Interval matrix, regularity, sufficient condition.

[1] Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

1 Introduction

State-of-the-art. Checking regularity of interval matrices is an NP-hard problem. At least forty necessary and sufficient conditions are at hand [5], but none of them can be used for practical computations due to their inherent exponentiality. Only two practically useful sufficient conditions exist (namely those by Beeck and Rump quoted below), which is in striking contradiction with the number of necessary and sufficient ones.

In this report we introduce the third sufficient condition, and we merge all the conditions into a single one. Then we demonstrate the mutual independence of the three conditions on examples, and we show how the main result can be applied to solvability of absolute value equations.

2 The condition

The new merged triple condition is formulated as follows.

Theorem 1. *Let A_c be nonsingular and let*

$$\min\left\{\varrho(|A_c^{-1}|\Delta),\ \varrho(|(A_c^T A_c)^{-1}|\Delta^T\Delta),\ \sigma_{\max}(\Delta)/\sigma_{\min}(A_c)\right\} < 1 \tag{2.1}$$

hold. Then the interval matrix $[A_c - \Delta, A_c + \Delta]$ is regular.

Proof. Obviously, (2.1) is satisfied if and only if either

$$\varrho(|A_c^{-1}|\Delta) < 1, \tag{2.2}$$

or

$$\varrho(|(A_c^T A_c)^{-1}|\Delta^T\Delta) < 1, \tag{2.3}$$

or

$$\sigma_{\max}(\Delta)/\sigma_{\min}(A_c) < 1 \tag{2.4}$$

holds. The inequalities (2.2) and (2.4) are the sufficient regularity conditions by Beeck [2] and Rump [7], respectively. So we are left with explaining the condition (2.3). If it holds, then by Beeck's condition the interval matrix $[A_c^T A_c - \Delta^T\Delta,\ A_c^T A_c + \Delta^T\Delta]$ is regular, which in turn implies that $[A_c - \Delta, A_c + \Delta]$ is regular (Farhadsefat, Lotfi and Rohn [3, Thm. 4.8]). □

3 Examples

For the purposes of this section denote

$$\begin{aligned} r &= \varrho(|(A_c^T A_c)^{-1}|\Delta^T\Delta), \\ s &= \varrho(|A_c^{-1}|\Delta), \\ t &= \sigma_{\max}(\Delta)/\sigma_{\min}(A_c). \end{aligned}$$

We shall demonstrate that for each $p, q \in \{r, s, t\}$, $p \neq q$, there exists an example with $p < 1 < q$, so that the condition p works whereas q does not. This will show that the three conditions are mutually independent, which means that none of them can be deleted from (2.1) without affecting its strength.

Example 1. Here $r < 1 < s$:

```
intval A =
[    0.1711,    0.8297] [    0.5806,    1.2756]
[    0.0911,    0.5707] [   -1.0011,   -0.5663]
r =
    0.7635
s =
    1.0092
```

Example 2. Here $s < 1 < r$:

```
intval A =
[   -0.8927,   -0.2063] [   -0.6980,   -0.4368]
[   -0.9232,   -0.3422] [   -0.1561,   -0.1351]
s =
    0.9582
r =
    1.5330
```

Example 3. Here $r < 1 < t$:

```
intval A =
[    0.1711,    0.8297] [    0.5806,    1.2756]
[    0.0911,    0.5707] [   -1.0011,   -0.5663]
r =
    0.7635
t =
    1.0154
```

Example 4. Here $t < 1 < r$:

```
intval A =
[   -0.2657,   -0.0638] [   -0.5740,   -0.4286] [    0.6923,    0.8241]
[   -0.5557,   -0.4718] [   -0.2749,    0.0053] [    0.7399,    0.8179]
[    0.2200,    0.5996] [    0.2508,    0.5727] [   -0.2336,    0.0517]
t =
    0.9680
r =
    1.0186
```

Example 5. Here $s < 1 < t$:

```
intval A =
[   -0.8927,   -0.2063] [   -0.6980,   -0.4368]
[   -0.9232,   -0.3422] [   -0.1561,   -0.1351]
s =
    0.9582
t =
    1.6280
```

Example 6. Here $t < 1 < s$:

```
intval A =
[   -0.5111,   -0.4952] [   -0.5671,    0.0993]
[   -0.0054,    0.5608] [   -0.4981,   -0.1923]
t =
    0.9692
s =
    1.1529
```

To assess the strength of the triple condition (2.1), we wrote a MATLAB file `triplesuffconds.m` (listed below) and then we ran it on 10,000 randomly constructed interval matrices of sizes varying between 2 and 10:

```
>> [suff,nsreg,nssng]=triplesuffconds(10,10000)
suff =
        3102
nsreg =
        3631
nssng =
        6369
```

Of the 10,000 matrices, 3,631 were found regular and 6,369 singular by a general, but sometimes slow file `regsing.m` (notice that it did not fail in a single case!), and 3,102 of these 3,631 regular matrices were found regular by our triple sufficient condition. At the first glance it looks like a very nice ratio; but the truth is that the result depends heavily on the way in which the random interval matrices are generated. For instance, if we generate in the same way 100×100 matrices, then the result is

```
>> [suff,nsreg,nssng]=triplesuffconds(100,10)
suff =
     0
nsreg =
     0
nssng =
    10
```

so that no regular interval matrix has been generated; this is because the radii of the generated matrices are too big in this case.

```
function [suff,nsreg,nssng]=triplesuffconds(m,j)
% m maximum size, j number of examples
suff=0; nsreg=0; nssng=0;
for i=1:j
    [A,rst,reg,sng]=randsizeintmat(m,i);
    if rst==1, suff=suff+1; end
    if reg==1, nsreg=nsreg+1; end
    if sng==1, nssng=nssng+1; end
end
%
function [A,rst,reg,sng]=randsizeintmat(m,i)
% A is the ith random square interval matrix, 2 <= size <= m
rst=0; reg=0; sng=0;
rand('state',i);
n=2+round(rand(1)*(m-2));
Ac=2*rand(n,n)-1;
Delta=0.1*rand(n,n);
A=midrad(Ac,Delta);
r=rho(abs(inv(Ac'*Ac))*Delta'*Delta);
s=rho(abs(inv(Ac))*Delta);
t=max(svd(Delta))/min(svd(Ac));
if min([r s t])<1, rst=1; end
[S]=regsing(A);
if  isempty(S), reg=1; end
if ~isempty(S), sng=1; end
%
function rh=rho(A) % spectral radius
rh=max(abs(eig(A)));
```

4 Application: Unique solvability of absolute value equations

Finally we show how our triple sufficient condition for regularity of interval matrices brings about a triple sufficient condition for unique solvability of an absolute value equation

$$Ax - |x| = b. \tag{4.1}$$

Theorem 2. *Let A be nonsingular and let*

$$\min\left\{\varrho(|A^{-1}|),\ \varrho(|(A^T A)^{-1}|),\ \sigma_{\max}(A^{-1})\right\} < 1 \tag{4.2}$$

hold. Then the equation (4.1) has a unique solution for each right-hand side b.

Proof. In the light of Theorem 1, the condition (4.2) implies regularity of the interval matrix $[A - I, A + I]$. The assertion then follows from [4, Prop. 4.2]. □

The condition (4.2) is a generalization of the condition

$$\min\left\{\varrho(|A^{-1}|),\ \sigma_{\max}(A^{-1})\right\} < 1 \tag{4.3}$$

which appeared in [6].

5 Acknowledgment

The work was supported with institutional support RVO:67985807.

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